NATURAL PHILOSOPHY

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PLAN OF NATURAL PHILOSOPHY

FOR

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NATURAL PHILOSOPHY

FOR

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AUTHOR'S NOTE.

THE writer has to express his indebtedness to many authors of other works on Natural Philosophy, notably R. Wormell, M.A., D.Sc.; Professor Newth, D.D.; and Philip Magnus, B.A., B.Sc. His friend Mrs. Annie Besant has also been of invaluable assistance in working out the examples.

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ERRATA.

Page 52, line 5, insert after 'supports,' 'and whose C. G. is 7 centimetres from its middle point.'

Page 98, line 8, delete 'of radius $1\frac{5}{44}$ dm.'

Page 239, line 10 from end, for '1 metre' read '1 decimetre'

NATURAL PHILOSOPHY

FOR

LONDON UNIVERSITY MATRICULATION EXAMINATION.

INTRODUCTION.

It is proposed in the following pages to submit to students a brief treatise upon Physics or Natural Philosophy. The writer is conscious of the fact, that not a few excellent treatises upon this subject are already in existence, some dealing with the science of Physics, even unto its most minute details, others treating the subject with the avowed design of presenting only its rudiments to the student, and assisting his passage over the threshold of the study. But it has seemed that there is yet room for another work upon Natural Philosophy, and an attempt will be made in these pages, however unworthily, to supply that which is required.

In the first place, the work essayed in these pages will be that which is required at the Matriculation of London University. Whilst the topics mentioned in the syllabus of that examination under the head Natural Philosophy will all be studied, the precise order wherein they are named in that syllabus will not be followed. The same ground as that covered by the phrases of that syllabus will be traversed, but the subjects will not be taken up in exactly the same succession as that encountered in the Calendar of our youngest University. An attempt will be made to lead the student gradually from simpler and more obvious facts to those more complex and less easily recognisable. The endeavour will be made to show some kind of connection between the different branches of the subject that are studied, and to indicate how the investigation of any one leads naturally to the examination of its successor.

But of far greater moment than any attempt to follow the lines of any particular examination of any particular University, is the design to teach any given subject in such fashion that the acquisition of special knowledge may fit its acquirer more completely for life as a whole. The acquisition of a mass of facts about animals or plants or minerals, the learning of a list of names, are of little value. It is the observation, the use of eyes, ears, hands, the mental effort, the employment of judgment, memory, reason, that are of use. The study of any particular subject is valuable in proportion as it strengthens the mental powers, and fits the mind more completely for accurate decision and ready action upon every occasion in life. This indeed suggests the answer to the ever-recurring cry of parents and of pupils, 'What's the good of learning this?' To those who thoughtlessly utter that cry, demonstration should be afforded of the fact that every fresh mental effort means greater ease of mind-movement afterwards; and that the use of our powers in the acquisition of any new truth implies greater ease in their employment hereafter, and the probability of more effective results. The value of the acquisition of new facts, and of the employment of our observation in reasoning in new ways, bears not so much relation to the immediate work in hand as to the life-work in all its infinite variety that is to come after.

With the conviction as to the truth of this strong in my mind, I venture to make one or two suggestions to students as to the method of work. And it may here be stated that these suggestions, if of any value at all, will avail not only in the study of Physics, but in any study whatsoever. Let me start with two truisms. (1) That no book is worth reading that is not worth analyzing. (2) That the ideas enunciated by a teacher, either by voice or pen, are not thoroughly the ideas of the learner until they have been expressed again in his own words. If, therefore, that which I am about to write is in any sense useful, it will be worth the while of the student to make analysis thereof. If the facts and principles I design to teach are to become part of the intellectual stock-in-trade of those that read my words, it will be well, perhaps it will be essential, that those facts and principles should be expressed in the language of the students themselves. Towards the attainment of these two ends I venture to make the following suggestions. They are suggestions that have grown out of the work done by myself whilst a student in the past and in the present. The note-book upon whose pages is to be inscribed the analysis of any particular book, should have a column ruled off on the left-hand side of every other page. This column may occupy about one-fifth of the page. It is in its turn to be occupied by what may be called keynotes, that is, the most salient facts and principles presented by the part of the subject that is immediately under consideration. The body of the page will serve for the full explanation of these principal words and phrases in the student's own language. In attending to any lecture, or in reading a chapter of any book, the student should notice and should note the most prominent ideas brought forward. After the lecture or after the reading of the chapter, the words or phrases embodying these ideas should be entered in the column on the left-hand side of the page. As each is thus entered, endeavour should be made to write out in the body of the page an explanation, which should be at once clear, concise, and full, of the word or phrase employed. When this has been done, if possible entirely without assistance from book or teacher, comparison should be made of the statements of the student with those of the lecturer or the book. The former should be corrected or enlarged by aid of the latter. Thus will be formed a more or less accurate transcript of the things taught, in the language of the student himself.

The object of confining notes to every other page only of the book, is that there may be room afforded by the blank alternate pages for the large alterations sometimes needed, and especially for the addition of new matter bearing upon the same subject derived from the student's after-work, or from his reading of other books.

Of the advantages of this method of work the two most evident are: (1) That the student has to think out and express in his own phraseology every fresh thought presented to him. Note-taking no longer is a feverish scribbling down of as many of the lecturer's phrases as possible, and a mechanical copying out thereof later on. It is a process involving close attention during the lecture or the reading, and the exercise of memory and of thought during the fuller outworking of the notes in the quiet of the study. (2) An analysis of any subject thus prepared will be found to be of value as a test of knowledge in the after time. Let the student take up such a book, and covering the larger part of the page, read keynote after keynote. Let him, as he reads each of these, endeavour mentally to recall or actually to write out the substance of that which has been written as expansion of the word or phrase he is now investigating. If he fail, let him re-read his former note. If this process be repeated until he can at suggestion of these keynotes reproduce the substance of all that he has learnt, he may rest assured that he is fairly master of the subject. In the pages that follow, assistance will be afforded to such students as may desire to work upon this plan, by the furnishing at the end of each of the earlier chapters, at least, of such principal notes as would constitute a skeleton analysis of the chapter. At times, some of the head-notes thus given will be of such a nature as to encourage the student in reasoning for himself. Thus the statement of any general principle will be followed occasionally by one or more notes, whose full working out will involve the application of that principle. Once or twice it is possible that illustrations will be given of the crude results of the earlier attempts in connection with this method of analysis, and of the alterations that have been found necessary to be made. All this will not preclude the presentation to the reader of a number of exercises upon the different parts of the science as they are investigated.

In the course of each principal division of the subject typical examples are worked out fully for the benefit of the student.

After each principal division of the subject, certain exercises will be given for solution by the student. Finally, as an Appendix at the conclusion of the book, will be found a general set of examples upon every subject dealt with in the body of the volume. Together with these will be found in the Appendix many of the questions that have been given at Matriculation during the last few years.

In working examples, keep the following rules in mind :—

State the principle whereon you purpose working.
Make diagrams.
Use no formula unless you comprehend fully its significance.
Employ the unitary method.
Work neatly.

DEFINITION AND DIVISIONS.

From its earliest years a child is surrounded by a world of beauty and of mystery. Through its sense avenues it drinks in impressions of that world. It becomes conscious of touches, tastes, odours, sounds, and sights. Very early it learns to distinguish between the objects causing these impressions and its own consciousness of them, although that consciousness is probably a function of brain matter. In common phrase, it learns to distinguish between matter and mind. The general name of Matter is given to all things outside our consciousness, but acting thereupon. Another name, less precise, more indefinitely used, is at times applied to matter. That name is Nature. The Greek word for nature is φυσις (*phusis*); hence the general study of matter is called Physics. Again, φιλεω (*phileo*) is the Greek word for 'I love;' σοφια (*sophia*) is the Greek word for 'wisdom.' Hence is derived the word Philosophy, which etymologically is 'the love of wisdom.' That particular form of study that deals with matter or nature generally, is sometimes known as 'Natural Philosophy.' On the other hand, the study of consciousness, or that special property of one form of matter that was supposed in the past to be beyond matter, is named from the Greek μετα (*meta*), 'beyond,' Metaphysics.

Physics or natural philosophy is therefore the study of matter. It will be observed, on reference to the preceding paragraph, that stress was laid upon the fact that Physics was the *general* study of matter. Matter has many forms; hence many branches of natural science. Each of these branches has to do with a particular kind of matter, but Physics deals with matter as a whole. It does not study the nature of plants, or animals, or minerals, but of all. Within its records are found truths that hold of all known forms of existence, not of any special few. Thus, to say that certain rocks have been formed by the deposit of sediment from water, is to state a geological fact. To say that plants feed upon carbon dioxide, is to state a botanical fact. To say that animals feed upon organic matter is to state a zoological fact. But the statement that all bodies that are allowed to fall freely under the earth's attraction move over 4·9 metres in the first second is a physical fact.

The various forms of matter around us are sometimes noticed to be apparently in the same position for a length of time, sometimes to be in different positions at successive points of time. To speak after ordinary fashion, bodies are at rest or in motion, for Motion is change of position. I have written '*apparently* in the same position,' for it must not be forgotten that all bodies upon or close to the earth share in the movements of that planet.

A horse draws a carriage along a road; a man moves the handle of a pump up and down; steam expanding raises a piston; the heat of a fire makes an iron bar increase in size; a great orator sways a vast assembly, sets its members thinking, and the best of them doing. As causes of these several movements we observe the force exerted by the horse, the force exerted by the steam, the force exerted by the heat of the fire, and the force exerted by the orator. Force, then, is the cause of motion.

This is not always quite apparent on the surface. Thus, a man exerts force by pushing against a mountain. He does not succeed in moving the mountain apparently. But in reality his force does produce movement, for his body and the mountain at the point of contact become warm. Heat is evolved, and heat is a form of movement—not of the palpable movement of distinct masses (that is, the only form of motion recognised by ordinary observers), but of the minute particles or little masses that are themselves as unseeable as their movements. That which all ordinarily call Motion is the movement of masses; heat is the movement of little masses. *Moles* is the Latin word for a mass, *icula* is a Latin diminutive termination; hence the distinction between molar movement and molecular movement. Molar movement is that which is usually regarded as movement, *e.g.* the movement of a cricket ball under the force of the batsman, the movement of a stone as it falls earthward under the force of gravitation. Molecular movement comprises those subtle position-changes of minute particles, whose results we call heat, or electrical, or magnetic, or vital phænomena. The form of motion especially to be studied in these pages is molar.

For convenience of work, it is well to have some simple method of representing forces. (1) Numerically they are represented by numbers of grams, as the gram is the unit of weight. This method is selected because most forces can be represented by weight, or can have weight substituted for them. Thus, the piston of the steam-engine might be drawn up by means of a cord passing over a fixed pulley, and having a *weight* attached to the free end. The steam is said to exert a pressure of say 100 kilograms upon the piston. That is to say, that the steam forces the piston upwards as rapidly as a weight of 100 kilograms would draw the piston up if connected with it by means of a cord and a fixed pulley. (2) Graphically, forces are represented by arrows. An arrow represents the direction of the force by its line of direction, the intensity of the force by its length, the point of application of the force by its point, whilst the arrow-head denotes whether the force be a pushing or a pulling one.

If one force only act on a body, molar movement must ensue in the line of direction of the force. If two or more forces act upon a body, molar movement may not ensue. If such molar movement result, the forces are said not to be in equilibrium; and the study of forces not in equilibrium is called Dynamics, from δυναμις (*dunamis*), force. If such molar movement do not result, the forces are said to be in equilibrium, and the study of forces in equilibrium is called Statics, from στασις (*stasis*), standing.

The first part of our book will be devoted to the study of Statics: the second, to that of Dynamics.

STATICS.

Two divisions of Statics present themselves—The Statics of Solids and the Statics of Liquids.

THE STATICS OF SOLIDS.

We shall consider—I. Forces acting upon a point; II. Forces acting upon a body.

I. Forces acting upon a point.

We shall consider—A. The action of one force; B. The action of two forces; C. The action of three forces; D. Of more than three forces.

CHAPTER I.

ACTION OF ONE FORCE AND OF TWO FORCES.

A. One force.—We have seen that molar movement must result in the direction of the force.

B. Two forces—(1) *In equilibrium.*—The only possible conditions under which two forces acting upon a body can produce no molar movement are the complete equality and the exact opposition of the two forces; they must be equal and opposite. (2) *Not in equilibrium.*—Three cases may occur: (*a*) where the two are parallel and acting in the same direction; (*b*) where they are parallel and acting in opposite directions; (*c*) where the two act at an angle of less than 180°.

In case (*a*), if one force had to be substituted for the two, it would be equal to their sum. Thus, if forces represented by 5 grams and 4 grams are acting parallel to one another and in the same direction upon a point, the one force that could be substituted for them and produce the same effect as they, would be a force represented by $5+4$ or 9 grams. This is evidently the greatest possible effect that could be produced.

In case (*b*), if one force had to be substituted for the two, it would be equal to their difference. Thus, if forces represented by 5 grams and 4 grams are acting parallel to one another, but in the opposite direction, upon a point, the one force that could be substituted for them and produce the same effect as they, would be a force represented by $5-4$, or 1 gram. This is evidently the least possible effect that could be produced. The single force that can be substituted for two or more others, and produce the same effect as they, is called their Resultant. Thus, in case (*a*), 9 represents the resultant of 5 and 4, and in case (*b*) 1 represents the resultant of 5 and 4. The forces for which the particular resultant can be substituted, are called in respect to it Components. Thus, in the two cases just given, 5 and 4 are the components of 9 and 1 respectively.

In case (*c*), where the two forces act at an angle of less than 180°, they are never acting parallel to each other and in the same direction, they are never acting parallel to each other and in the opposite direction; therefore their resultant can never be so much as their sum, and it can never be so little as their difference. It will always be between their sum and their difference. In fact, the forces are never working together, but always pulling to some extent against each other; therefore the resultant cannot be the greatest possible. Also the forces are never working in exactly opposite directions, but are, to some extent, pulling one with the other; therefore the resultant cannot be the least possible. The smaller the angle between the two forces, the closer is the approach towards the state where they are parallel and acting in the same direction; therefore the greater is the resultant. The larger the angle between them, the closer is the approach towards the state where the forces are parallel and acting in opposite directions; therefore the less is the resultant. The method of obtaining the value of this resultant exactly, both graphically and numerically, will be shown in the next pages. I will now proceed to sum up the matter of the preceding pages in the form of certain keynotes. These, as explained above, should be entered in the small columns on the left-hand side of the note-book, and concise but sufficient explanation of them given by the student in the body of the page:—(1) Matter; (2) mind; (3) physics; (4) metaphysics; (5) exact scope of physics; (6) motion; (7) the two kinds; (8) examples; (9) force; (10) numerical representation of force; (11) graphic representation of force; (12) equilibrium; (13) statics; (14) dynamics; (15) action of one force; (16) of two, equal and opposite, acting on the same point; (17) of two parallel forces; (18) resultant; (19) components; (20) the greatest and the least possible resultant; (21) two forces at an angle of less than 180°.

Example 1.—Three forces represented by 4, 10, and 6 grams respectively, act on a point in the same direction. On the same point act two forces of 5 and 7 grams. Find what force must be introduced to keep the point at rest.

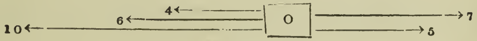


Fig. 1.

It is impossible to represent a point accurately. Let O stand for a point. The resultant of the forces 4, 6, $10 = 4 + 6 + 10 = 20$. The resultant of the forces 5 and $7 = 5 + 7 = 12$.

The resultant of all the forces $= 20 - 12 = 8$ in the direction of the three forces. To keep the point at rest, a force equal and opposite to this must be introduced, *i.e.* a force $= 8$ acting in the same direction as the 5 and 7.

Example 2.—Find the value of four forces all acting in the same direction when their values are in arithmetical progression (common difference $= 2$), and they are balanced by three forces each equal to the largest of the four.

Let $x =$ value of largest force.

Then $x - 2$, $x - 4$, $x - 6 =$ values of other three.

$$x + (x - 2) + (x - 4) + (x - 6) = 3x.$$
$$4x - 12 = 3x.$$
$$x = 12.$$

Forces are 12, 10, 8, 6, and $12 + 10 + 8 + 6 = 36$, *i.e.* 3 times 12, the largest force.

EXERCISES.

1. Give a fact in relation to matter not strictly coming under the domain of Physics, and two facts in relation to matter that are included in that domain.

2. An instance of the conversion of molar into molecular motion, and an instance of the converse.

3. Draw a representation of—(*a*) one force acting on a point; (*b*) two forces dragging upon a point; (*c*) one force pushing and one force pulling, producing the least possible effect; (*d*) one force pushing and one force pulling, producing the greatest possible effect.

4. Draw to scale, forces of 2, 5, 11, and 1 grams respectively, all acting in the same direction; calculate their resultant, and find what must be added to a force of 7, acting in the opposite direction, to keep the point at rest.

5. Acting upon a point are forces 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, all the even numbered forces acting in one direction, the odd numbered in the opposite. Find the magnitude and direction of the resultant.

6. Acting on a point in the same direction are forces of 4, 17, -12, in the opposite direction are forces of 8, 1, 9, -9. Calculate magnitude and direction of resultant.

7. If the resultant of two forces acting parallel and in the same direction is 28, and one component is 19, find the other.

Under the same conditions, save that the forces are acting in opposite directions, find the value of the other component.

✓ 8. Four forces in arithmetical progression act parallel to one another and in the same direction. The least force is 2 grams, and the common difference 3. What number of forces on the opposite side will keep the point at rest, if they are also in arithmetical progression, with a common difference of two, and are aided by a force equal to the weakest of them (viz. 1) working on the same side as, but against, the first set of forces? Find also the value of each of the forces.

✓ 9. What must be the weakest of a set of five forces, that if they are all acting parallel and in the same direction, with a common ratio of 2, they may exactly balance two forces of 100 and -7 acting parallel to them but in the opposite direction?

10. A circular disc is pushed upon by four forces whose lines of action if produced would meet in the centre of the disc at right angles to each other. The forces taken in order are 4, 7, -3, 12. By the introduction of what forces would you keep the disc at rest?

All worked.

CHAPTER II.

PARALLELOGRAM OF FORCES.

Action of two forces that are (2) not in equilibrium and (*c*) act at an angle of less than 180°. The resultant has been shown to be less in value than the sum of the two forces, greater in value than their difference. It now remains to find a method of determining the exact value of the quantity that represents the resultant, and that lies between the sum and the difference of the numbers representing the two forces. This is effected by means of the theorem known as the Parallelogram of Forces. *If two forces acting upon a point are represented in magnitude and direction by the two adjacent sides of a parallelogram, the diameter of that parallelogram drawn through the point will represent their resultant in magnitude and direction.* Expressing this abstract theorem, which admits of easy practical proof, in more concrete form: if two forces P and Q, in Fig. 2, acting upon a point O, are represented in magnitude and direction by the two adjacent sides OA, OB of the parallelogram OBCA, the diameter OC of the parallelogram drawn through the point O will represent their resultant R in magnitude and direction.

The *direction* of the resultant can then always be determined by drawing, *i.e.* graphically; its magnitude can also be obtained either approximately in every case by the graphic method, or exactly, when the forces act at certain angles, by calculation.

(i.) Graphically.—Suppose two forces, represented by 4 grams and 3 grams respectively, act upon a point, and that their directions are inclined to each other at right angles. It is required to find their resultant. Take any point O, Fig. 2. From O draw a horizontal line. Take some unit of length (say $\frac{1}{2}$ inch) to represent 1 gram. Mark off along this line four such units (say 2 inches) to represent 4 grams. The line OA represents the force of 4 grams. Draw from O a vertical line. Mark off along this line three units of length to represent 3 grams. The line OB represents the force of 3 grams. Through A draw a line parallel to OB. Through B draw a line parallel to OA. Thus is formed a

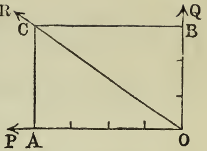


Fig. 2.

parallelogram OBCA. Join OC. This line will represent the resultant of the two given forces, and if measured will be found to contain the unit of length five times. Therefore the resultant of two forces of 4 and 3 grams respectively, acting at an angle of 90° to each other, is graphically determined as 5 grams. Similarly, by a careful drawing to scale, the value of the resultant of any two forces acting at any angle can be approximately determined.

(ii) By calculation.—Without the aid of Trigonometry, it is possible to calculate accurately the numerical value of the resultant of forces acting at angles of 90°, 45°, 60°, 30°, 135°, 150°, 120°.

(α) Two forces acting at 90°.—Consider Fig. 1. It is required to determine the value of R, the resultant of P and Q, two forces acting upon the point O at an angle of 90° with each other. R is represented by the line OC. P is represented by the line OA. Q is represented by the line OB = AC. In the triangle OAC, square upon OC = sum of squares upon OA and AC; $\therefore$ $OC^2 = OA^2 + AC^2$; $\therefore$ $R^2 = P^2 + Q^2$; $\therefore$ $R^2 = 4^2 + 3^2 = 16 + 9 = 25$, and $R = \sqrt{25} = 5$. The value of the resultant of two forces acting at an angle of 90° with each other, can be found by the application of the proposition that in any right-angled triangle the square on the side opposite the right angle = the sum of the squares on the sides containing the right angle.

(β) Two forces acting at an angle of 45°.—Draw an isosceles right-angled triangle AOB, with O as the right angle. The angles are therefore 90° at O, 45° at A, 45° at B. It will be well to determine once for all the ratio of the lengths of the three sides AB, BO, OA to one another in a triangle whose angles are 90°, 45°, 45°. Denote the side AB opposite the right angle by a. Then, as the triangle is right-angled, a^2 = the sum of the squares on the other two sides = twice the square on one of them $= 2x^2$; $\therefore$ $x^2 = \frac{a^2}{2}$; $\therefore$ $x = \frac{a}{\sqrt{2}}$. Multiply this last fraction by $\frac{\sqrt{2}}{\sqrt{2}}$ or 1, and x, the value of either of the sides containing the right angle $= \frac{a\sqrt{2}}{2}$.

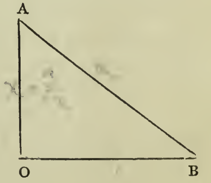


Fig. 3.

Hence, in any triangle whose angles are respectively 90°, 45°, 45°, the ratio of the side opposite to the right angle to either of those containing the right angle is as $a : \frac{a\sqrt{2}}{2}$.

(γ) Two forces acting upon a point at angles of 60° or 30°. Draw an equilateral triangle DEI. Bisect $\angle$ at D, and join with the middle point T of base EI. Consider triangle ETD. $\angle$ETD is 90°, $\angle$DET = 60°, $\angle$EDT = 30°. It is required to establish a ratio between the sides of a triangle whose angles are respectively 90°, 60°, 30°. Let the side ED opposite the angle of 90° be represented by a. ET, the side opposite 30° $= \frac{1}{2}$ED $= \frac{a}{2}$.

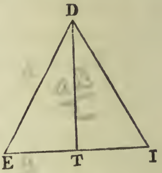


Fig. 4.

To find TD, the side opposite 60°, let DT be represented by x. Then $ED^2 = ET^2 + DT^2$;

$$\therefore\ a^2 = \frac{a^2}{4} + x^2\ ;\ \therefore\ x^2 = \frac{3a^2}{4}\ ;\ \therefore\ x = \frac{a\sqrt{3}}{2}.$$

That is, in a triangle whose angles = 90°, 30°, 60° respectively, the sides opposite the angles 90°, 30°, 60° are in the ratio of a, $\frac{a}{2}$, $\frac{a\sqrt{3}}{2}$.

By the aid of (α), (β), or (γ), any ordinary question in respect to the parallelogram of forces can be solved. We have seen the methods of solving the cases where the angle between the two forces = 90°, 45°, 30°, or 60°. Combine 90° with 45°, with 30°, with 60°, and we have forces acting at angles of 135°, 120°, 150°. One of each of the typical cases of forces acting at these special angles will now be studied, and the parallelogram of forces will then be considered as sufficiently worked out for matriculation purposes.

1. Angle = 30°.

Let P and Q be two forces acting upon a point at an angle of 30°. If P is represented by 12 grams, and R the resultant of the two forces by 16 grams: find the value of Q.

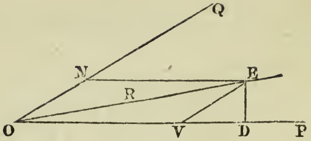


Fig. 5.

Let O represent the point acted upon. Mark off along the line of P, OV to represent 12. Draw through V a line parallel to direction of Q. Take O as centre, and with radius = 16 on the same scale as OV was taken, describe a circle. From where this circle cuts the parallel to direction of Q, draw EN parallel to P. ON represents the value of Q. From E draw ED perpendicular to P. Consider triangle VED. $\angle EDV = 90°$, $\angle EVD = 30°$, $\angle VED = 60°$;

$$\therefore \text{ sides VE, DE, VD are as } a, \frac{a}{2}, \frac{a\sqrt{3}}{2}. \quad \text{But VE} = x;$$

$$\therefore DE = \frac{x}{2}, \text{ and } VD = \frac{x\sqrt{3}}{2}.$$

Draw OE representing 16.

Consider triangle OED. $OE^2 = OD^2 + DE^2 = (OV + VD)^2 + DE^2$;

$$\therefore 16^2 = \left(12 + \frac{x\sqrt{3}}{2}\right)^2 + \left(\frac{x}{2}\right)^2 = 144 + 12x\sqrt{3} + \frac{3x^2}{4} + \frac{x^2}{4}.$$

Hence work out value of x.

2. Angle = 45°.

Let P and Q be two forces acting at an angle of 45° upon a point. It is required to find their resultant if P = 9 and Q : P :: 1 : 3.

Let O represent the point acted upon. Mark off along the line of P, OG to represent 9. By the question, P = 3Q; $\therefore Q = \frac{P}{3}$ = 3. Mark off along the line of Q, OK = 3. Complete the parallelogram OKIG. Draw the diameter OI through the point O.

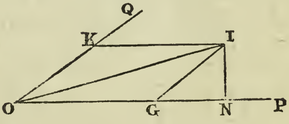


Fig. 6.

OI will represent the resultant of the two forces P and Q.

Produce OG. From I draw IN perpendicular to OG produced. Consider triangle GIN. $\angle N = 90°$. Angles NGI, NIG each = 45°. Sides GI, IN, NG have ratios $a, \frac{a\sqrt{2}}{2}, \frac{a\sqrt{2}}{2}$.

But GI = 3; ∴ IN and NG each $= \frac{3\sqrt{2}}{2}$. Consider triangle OIN. $OI^2 = ON^2 + NI^2$; $\therefore x^2 = (OG + GN)^2 + NI^2 =$

$\left(9+\frac{3\sqrt{2}}{2}\right)^2+\left(\frac{3\sqrt{2}}{2}\right)^2=81+27\sqrt{2}+\frac{9}{2}+\frac{9}{2}$. Hence work out value of x.

3. Angle = 60°.

Let $P=10$ and $Q=7$ be two forces acting upon the point O at an angle of 60°. It is required to find their resultant R. Along direction of P mark off OA representing 10. Along direction of Q mark off OD representing 7. Through D draw DB parallel to OA, and through A draw AB parallel to OD. Thus a parallelogram ODBA is formed. Through the point O draw the diameter OB. This will represent the resultant of P and Q in direction and in magnitude. It is required to find the value of OB. From B let fall a perpendicular upon OA produced. Then BAI is a right-angled triangle. The angle at I = 90°, that at A = 60°, that at B = 30°; ∴ sides BA, AI, IB are as a, $\frac{a}{2}$, $\frac{a\sqrt{3}}{2}$. But $BA=7$; ∴ $AI=3\frac{1}{2}$, and $IB=3\frac{1}{2}\sqrt{3}$. Now consider triangle OBI. $OB^2=OI^2+IB^2$; ∴ $x^2=(10+3\frac{1}{2})^2+(3\frac{1}{2}\sqrt{3})^2=(13\frac{1}{2})^2+\frac{49}{4}\times 3$. Hence work out value of x.

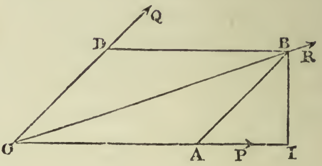


Fig. 7.

4. Angle = 120° (90° + 30°).

Let P and Q be two forces acting upon a point at an angle of 120°, having ratio to each other of 7 to 4. Their resultant = 12·5 grams. It is required to find the value of P and of Q.

$\frac{P}{Q}=\frac{7}{4}$; ∴ $4P=7Q$; ∴ $P=\frac{7Q}{4}$.

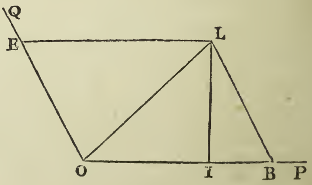


Fig. 8.

Let O represent the point acted upon. Mark off along the line of P, OB to represent 7, and along the line of Q, OE to represent 4. Complete parallelogram OELB. Draw diameter OL through the point O. OL represents 12·5. From L draw LI perpendicular to P. Consider triangle LIB. ∠LIB = 90°,

$\angle$ILB $= 30°$, $\angle$LBI $= 60°$; $\therefore$ sides LB, BI, IL are as $a, \frac{a}{2}, \frac{a\sqrt{3}}{2}$. But LB represents Q; $\therefore$ BI represents $\frac{Q}{2}$, and IL represents $\frac{Q\sqrt{3}}{2}$. Consider triangle OLI. $OL^2 = OI^2 + IL^2$; $\therefore (12{\cdot}5)^2 = OI^2 + IL^2 = (OB - IB)^2 + IL^2 = \left(P - \frac{Q}{2}\right)^2 + \left(\frac{Q\sqrt{3}}{2}\right)^2 = \left(\frac{7Q}{4} - \frac{Q}{2}\right)^2 + \left(\frac{Q\sqrt{3}}{2}\right)^2 = \left(\frac{5Q}{4}\right)^2 + \frac{3Q^2}{4} = \frac{25Q^2}{16} + \frac{3Q^2}{4}$. Hence work out value of Q, and thence value of P.

5. Angle $= 135°$ $(90° + 45°)$.

Let P and Q be two forces acting at an angle of 135°. It is required to find their resultant if $P = 6$ and $Q = 2P$.

Let O represent the point acted upon. Mark off along P, $OA = 6$, and along Q, $OT = 12$. Complete the parallelogram OTLA. Draw the diagonal OL through the point O. OL will represent the resultant of the two forces P and Q. From L draw LN perpendicular to OT. Consider the triangle LNT. $\angle$LNT $= 90°$, $\angle$NTL $= 45°$, $\angle$TLN $= 45°$; $\therefore$ sides TL, LN, NT are as $a, \frac{a\sqrt{2}}{2}, \frac{a\sqrt{2}}{2}$. But $TL = 6$; $\therefore$ LN and NT = respectively $3\sqrt{2}$. Now consider triangle LNO. $OL^2 = ON^2 + NL^2$; $\therefore x^2 = (OT - NT)^2 + NL^2$; $\therefore x^2 = (12 - 3\sqrt{2})^2 + (3\sqrt{2})^2 = 144 - 72\sqrt{2} + 18 + 18$. Hence work out value of x.

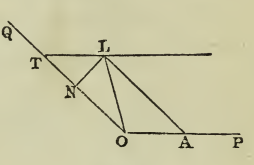


Fig. 9.

6. Angle $= 150°$ $(90° + 60°)$.

Let P and Q be two forces acting upon a point at an angle of 150°, $P = 11$, $Q = 9$. It is required to find the value of their resultant.

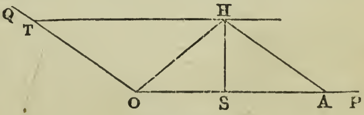


Fig. 10.

Let O represent the point acted upon. Mark off along the line of P, OA to represent 11; along the line of Q, OT to represent 9. Complete the parallelogram, and through the point O draw diameter OH. From H draw HS perpendicular to P. Consider triangle HSA.

$\angle HSA = 90^\circ$, $\angle HAS = 30^\circ$, $\angle SHA = 60^\circ$; $\therefore$ sides HA, HS, SA are as $a, \frac{a}{2}, \frac{a\sqrt{3}}{2}$. But $HA = 9$; $\therefore$ $HS = 4\frac{1}{2}$, $SA = 4\frac{1}{2}\sqrt{3}$. Consider triangle HSO. $OH^2 = OS^2 + SH^2$; $\therefore OH^2 = (OA - SA)^2 + SH^2 = (11 - 4\frac{1}{2}\sqrt{3})^2 + (4\frac{1}{2})^2 = 121 - 99\sqrt{3} + \frac{243}{4} + \frac{81}{4}$. Hence work out value of OH which represents the resultant.

Key-notes to the above are:—(1) Parallelogram of forces as an abstract theorem; (2) parallelogram of forces stated in concrete form; (3) graphic method of finding value of resultant of two forces; (4) the relationship of the square on the hypotenuse to the square on the sides of a right-angled triangle; (5) proof that in a triangle whose angles are 90°, 45°, 45°, the sides are as $a, \frac{a\sqrt{2}}{2}, \frac{a\sqrt{2}}{2}$; (6) that in a triangle whose angles are 90°, 30°, 60°, the sides are as $a, \frac{a}{2}, \frac{a\sqrt{3}}{2}$; (7) diagram of two forces acting at an angle of 90° and of their resultant; (8) the same with angle = 45°; (9) the same with angle = 30°; (10) = 60°; (11) = 135°; (12) = 120°; (13) = 150°.

EXERCISES.

1. Two forces at angle 90°. One = $\frac{4}{5}$ of the other. Resultant = $\sqrt{656}$. Find value of forces.

2. Two forces at angle 90° are in the ratio of 11 to 7. The lesser force = 21 grams. Find value of the greater and of the resultant.

3. Two equal forces at angle of 45° have a resultant = 20 grams. Find forces.

4. Compare the resultants of two forces of 4 and 3 grams respectively acting at angles of 90°, 30°, 150°, 180°. ✓

5. The resultant of two equal forces acting at an angle of 60° is equal to the resultant of two forces of 12 grams and 8 grams respectively at angle 45°. Find the two forces.

6. The resultant of two forces acting at an angle of 30° is twice one of the components. Find its ratio to the other component.

Under key-note (5), page 13, a student writes:—'Physics does not deal with botany or zoology or geology, but with things generally.' A better note would have been:—'Physics deals with facts true of matter generally, not facts true of special forms of matter only.' Under key-note (15): 'To produce movement.' Better: 'One force only acting produces movement in the direction of the force.'

CHAPTER III.

TRIANGLE OF FORCES.

The action of one force upon a body, and the action of two forces upon a body, have been considered. C. The action of THREE FORCES. (1) *In equilibrium.* Suppose two forces P and Q acting upon a point O. Mark off along the line of direction of P, OL to represent the magnitude of P, and along the line of direction of Q, OE to represent the magnitude of Q. By means of the parallelogram of forces, find a line OF to represent the resultant R of these two forces. R will produce the same effect as P and Q will produce when acting together upon the point O. If only R were acting upon O, it would be possible to keep that point at rest by introducing another force exactly equal to R in magnitude, and diametrically opposed to R in direction. Let such a force be represented by the line OI, and be called R′. If R and R′ were acting upon the point O, that point would remain at rest as far as molar movement is concerned. But the effect of P and Q acting together equals that of R. ∴ if P, Q, R′ are acting upon the point O, that point will remain at rest as far as molar movement is concerned. P, Q, and R′ are therefore three forces in equilibrium. They are represented respectively in magnitude by the lines OL, OE, OI. Consider triangle OLF. Its sides OL, LF, FO are severally parallel to the directions of the three forces that are in equilibrium, and the lengths of those three sides are proportional also to the magnitudes of the three forces in equilibrium. This investigation leads us to the enunciation of a principle almost as important as that already stated under the name of the Parallelogram of Forces. This principle is the Triangle of Forces.

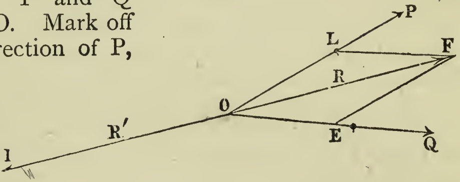


Fig. 11.

If three forces acting upon a point are in equilibrium, and any triangle be drawn whose three sides taken in order are severally *parallel to or perpendicular to the directions of the three forces, the lengths of those sides will be proportional to the magnitudes of the forces.* The converse of this holds. If three forces have magnitudes proportional to the lengths of the sides of any triangle that can be drawn in such manner that the directions of its three sides taken in order are parallel to the directions of the three forces, these forces are in equilibrium.

It will be observed that the sides of the triangle may be drawn *perpendicular* to the directions of the three forces, as well as parallel. As two straight lines perpendicular to two other straight lines contain an angle equal to that contained by the two lines to which they are perpendicular, the triangle formed by lines drawn at right angles to the directions of the three forces in equilibrium will be equiangular to the triangle formed by lines drawn parallel to the directions of the three forces in equilibrium, and its sides will bear the same proportion to one another as do the sides of the latter triangle.

In all cases, therefore, where three forces are in equilibrium, endeavour should be made to discover some triangle whose sides are severally parallel or perpendicular to the directions of the three forces, *and the relative lengths of the sides whereof are known.* Any triangle drawn at random, without knowledge of its angles or of the length of its sides, will not avail to determine the value of the forces.

Example 3–The ends of two cords are fastened to two points in the same horizontal line 5 decimetres apart. A smooth ring, weighing 20 grams, and supporting a weight of 30 grams, is attached to the other ends of the cords. It is found that the angle made by the two parts of the cord = 90°, and that one part of the cord is 3 decimetres long. Find the whole length of the two cords, and the tension in each of them.

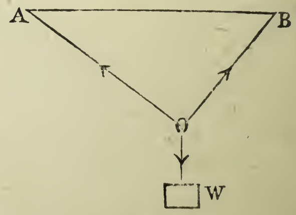


Fig. 12.

Let A, B represent the two points in the same horizontal line, 5 decimetres apart. Let O represent the ring weighing 20 grams, W the weight of 30 grams. $\angle AOB = 90°$. $OB = 3$ decimetres long. It is required to find the whole length of the cord AOB, and the tensions in OA, OB respectively.

(*a*) AOB = right-angled triangle ; $\therefore AB^2 = AO^2 + OB^2$; $\therefore 5^2 = AO^2 + 3^2$; $\therefore 25 = AO^2 + 9$; $\therefore AO^2 = 16$, and $AO = 4$ decimetres. Whole length $= AO + OB = 4 + 3 = 7$ decimetres.

(*b*) Three forces acting upon O are in equilibrium. They are—(i.) the weight of ring and weight, acting, as all weights act, vertically downwards ; (ii.) the tension along the cord OA ; (iii.) the tension along the cord OB. But these three forces are in equilibrium. Let us therefore search for a triangle whose sides, taken in order, are severally parallel or severally perpendicular to the directions of these three forces. Consider triangle AOB. Side AO is parallel to direction of tension along cord OA. Side BO is parallel to direction of tension along cord OB. But AB is *not* parallel to force due to gravitation acting vertically downwards. We cannot find a workable triangle with sides parallel to the directions of the three forces in equilibrium. But side AO of triangle AOB is perpendicular to and its length proportional to tension in OB. Side BO is perpendicular to and its length proportional to tension in OA. Side AB is perpendicular to force due to gravitation, *i.e.* to force = 50 grams, and its length is proportional to the magnitude of that force.

AB represents 50 grams.
AO ,, tension in OB.
BO ,, ,, OA.
5 ,, 50 grams.
1 ,, $\frac{50}{5}$ grams.
4 ,, $\frac{\overset{10}{\cancel{50}}}{\cancel{5}} \times 4 = 40$ grams.
3 ,, $\frac{\overset{10}{\cancel{50}}}{\cancel{5}} \times 3 = 30$,,

∴ tension in part of cord OB = 40 grams ;
,, ,, OA = 30 grams.

(2) *Three forces not in equilibrium.*—Find the resultant of any two of them, and compound that resultant with the third.

Example 4–Two forces represented by 10 grams and 6 grams respectively act upon a point at an angle of 180° with each other. A third, represented by 8 grams, acts upon the same point, and is inclined to the smaller of the two other forces at an angle of 60°. Find resultant.

P and Q represent two forces of 10 and 6 grams respectively, diametrically opposed to each other, and acting upon O; R, a third force, acting upon O, and at an angle of 60° to Q. First find resultant of P and $Q = 10 - 6 = 4$ acting along the line of P. Mark off along the line of P, OD to represent 4; along the line of R, OC to represent 8. By parallelogram of forces, find OE representing the resultant of the forces of 4 and 8 grams. Angle $COB = 60°$; ∴ angle $DOC = 120°$, and OE is the resultant of two forces represented by 4 and 8 grams respectively acting upon a point at an angle of 120°. Find this after the manner shown on page 20 (4).

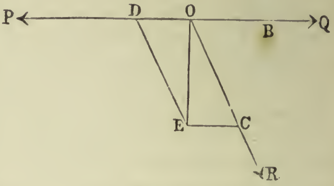


Fig. 13.

Example 5.—Three forces P, Q, R act upon a point O; angle between P and $R = 120°$; angle between P and $Q = 60°$, $P = 6$ grams, $Q = 6$ grams, $R = 12$ grams. Find resultant.

Along line of P mark off OA to represent 6 grams; along line of Q mark off OB to represent 6 grams; along line of R mark off OC to represent 12 grams. Compound OA, OB acting at angle of 60°. Find OD representing the resultant of P and Q; find its value after the manner shown on page 20 (3). Next compound OD and OC, acting at angle of 90° ($BOC = 60°$, $DOB = 30°$). Find OE representing resultant of P, Q, R. Find its value after manner shown on page 17 (α).

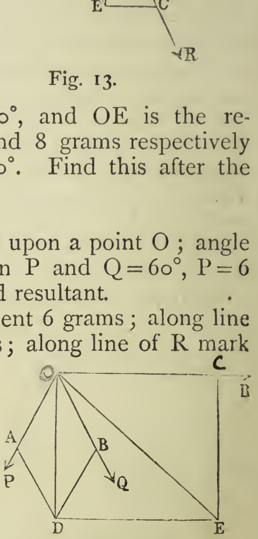


Fig. 14.

Exercises.

1. A string 1 metre long is fastened to two points in the same horizontal line. The two parts of the string make an angle of 90° with each other. A weight of 50 grams slides along the string. When it is at rest, determine the tension in the cord.

2. A ball weighing 10 grams slides without friction down a rod inclined at an angle of 30° with the vertical. A string is attached to it and stretched horizontally by the hand. Determine

the force exerted by the hand, and the pressure on the rod when the ball is at rest.

3. Over two points in the same horizontal line 6 decimetres apart, cords pass, and to each a weight of 4 grams is attached. The other extremities of the cords are tied to a weight. What are the lengths of the cords, and what is the weight supported by them when they make an angle of 90° with each other?

4. A ball sliding down a rod inclined at an angle of 45° with the vertical is kept up by a horizontal force $=12$ grams. What is the weight of the ball?

5. One end of a cord 4 decimetres long is attached to a weight of 20 grams, the other fixed to a point. Another cord 3 decimetres long is tied to the weight, and then fixed at the other extremity to a point on the same horizontal line as the former. The angle between the two cords $=90°$. Find the tension in each cord.

6. To a weight of 1 kilogram three cords are attached. One runs vertically upwards, and passing over a frictionless peg, has a weight of 2 hectograms affixed to its free extremity. Another, at an angle of 60° with the former, passes over a frictionless peg, and has a weight of 5 hectograms attached to it. What weight attached to the third will cause equilibrium, if the third make an angle of 30° with the first?

Under key-note (1), page 22, a student writes: 'If two forces acting upon a point are represented in magnitude and direction by two sides of a parallelogram, their resultant will be represented by the diameter of the parallelogram.' Two additions are needed here. Insert between 'two' and 'sides' the word 'adjacent'; after the second 'parallelogram,' 'drawn through the point.'

Under key-note (5), page 22, a student proves that the sides are as a, $\frac{a}{\sqrt{2}}$, $\frac{a}{\sqrt{2}}$; but does not express the last two quantities in the more convenient and workable form $\frac{a\sqrt{2}}{2}$, $\frac{a\sqrt{2}}{2}$, which is derived from $\frac{a}{\sqrt{2}}$ by multiplying numerator and denominator by $\sqrt{2}$ without any alteration of the value of the fraction.

CHAPTER IV.

MORE THAN THREE FORCES.

THE action of one force upon a body, the action of two forces upon a body, the action of three forces upon a body, have been considered.

D. THE ACTION OF MORE THAN THREE FORCES—(1) *In equilibrium.*—An extension of the principle of the triangle of forces comes into play. It is, when enunciated, known as the polygon of forces.

Five forces represented respectively by 3, 3, 3, $\sqrt{18}$, 6 grams, act upon a point. The angles between successive pairs are 120°, 30°, 45°, 135°. Show that the forces are in equilibrium.

Let the forces be represented by the lines OA, OB, OC, OD, OE, Fig. 15, and let them act upon a point O and be in equilibrium. Take any point O′, Fig. 16.

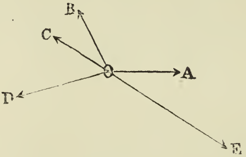


Fig. 15.

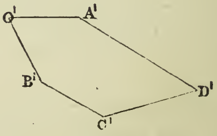


Fig. 16.

Draw O′A′ parallel and equal to OA. Through O′ draw O′B′ parallel and equal to OB. Through B′ draw B′C′ parallel and equal to OC. Through C′ draw C′D′ parallel and equal to OD. Through D′ draw a line parallel and equal to OE. This line will be found to pass through A′. A polygon will be formed.

This polygon is closed as the forces are in equilibrium, and their resultant as far as molar motion is concerned is nothing. And so generally if any number of forces acting upon a point are in equilibrium, and a polygon be drawn whose sides taken in order are parallel and proportional to the directions and magnitudes of the forces, the polygon will be a closed one.

(2) *Not in equilibrium.*—Find resultants of successive pairs of forces. Combine these in pairs until one final resultant is obtained.

Example 6.—Let us take the case last given, on the supposition that we do not know equilibrium would result. See Fig. 17.

(*a*) Combine OA and OB, equal forces of 3 grams at an angle of 120°. Resultant represented by OR (4, page 20) = 3.

(*b*) Combine OC and OE. $\angle AOC = 120^\circ + 30^\circ = 150^\circ$, $\angle AOE = 360^\circ - (120^\circ + 30^\circ + 45^\circ + 135^\circ) = 30^\circ$; $\therefore$ $\angle AOC + \angle AOE = 180^\circ$, and OC and OE are in the same straight line.

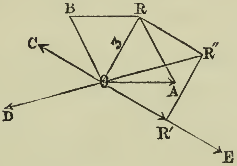


Fig. 17.

Their resultant $\therefore$ is equal to their difference, and acts in the direction of the greater OE = 3 represented by OR′.

(*c*) Combine OR and OR′. The angle between these = R′OA (30°) + AOR (60°) = 90°; $\therefore$ resultant of these two forces, each of 3 at an angle of $90^\circ = OR'' = \sqrt{(9+9)} = \sqrt{18}$.

(*d*) Combine OR″ and OD. $\angle$ between them $= \angle R''OR + \angle ROB + \angle BOC + \angle COD = 45^\circ + 60^\circ + 30^\circ + 45^\circ = 180^\circ$; $\therefore$ they are in the same straight line, and as they each $= \sqrt{18}$, they are equal and opposite, and equilibrium must ensue.

EXERCISES.

1. Three equal forces act on a point. The angle between two of them = 120°. The third bisects that angle. What force must be introduced to keep the point from moving, and what angles does its direction make with the directions of each of the three original forces?

2. On a point act four forces,—two at right angles to each other, each represented by $\sqrt{24 \cdot 5}$ grams; a third at an angle of 15° with one of these, and represented by 7 grams; a fourth at an angle of 150° with the third. What is the value of the last if the forces are in equilibrium?

3. Find resultant of $\sqrt{2}$, $\sqrt{2}$, 2, $\sqrt{12}$ acting on a point when the angles between successive pairs of forces are respectively 90°, 45°, 90°.

CHAPTER V.

II. Forces acting upon a Body.

A. MOMENTS.

Thus far we have considered the action of forces one, two, three, or many upon a point. We pass to the consideration of two forces acting not upon one point, but upon a body, and parallel to each other.

Parallel forces may act in the same direction or in opposed directions. We will consider the magnitude of their resultant, and the position of their resultant.

The magnitude of their resultant.—This will be equal to their algebraical sum. Let all the forces acting in one direction be regarded as positive, and all those acting in the opposite direction as negative. Sum the values of each set. Obtain the difference of these sums. This will be the magnitude of the resultant, and the direction will be in that of the set of forces whose sum is the greater.

The position of the resultant.—To determine this the principles of moments must be mastered.

A. Moments.

(1) Definition and Measurement.

If a rigid bar be held by the hand and a weight suspended at different points in the bar, the greater the distance between the point of suspension of the weight and the hand, the greater is the strain upon the hand. If from the same point different weights are hung, it is found that the greater the mass, the greater is the strain upon the hand.

The tendency of the masses to cause rotation of the rigid bar round the fixed point is called their moment, and generally *moment is the tendency of a force to cause rotation of a rigid body around a fixed point.* It has been shown that this tendency varies as the distance between the point of suspension of the mass and the fixed point, and as the mass; and generally moment is measured by the product of the number of units of force and the number of units of length in the perpendicular line drawn from the point to the line of direction of the force. This product is an abstract, not a concrete number, but the moments of different forces under different circumstances can be compared through the medium of the abstract numbers representing them.

Example 7.—A rigid bar 1 metre long has weights of 4 grams, 6 grams, 10 grams, hung respectively at the two ends and in the middle of the bar. Calculate the abstract value of their respective moments about a point 4 decimetres 5 centimetres from the end to which the 6 grams is attached.

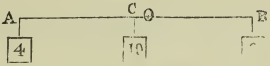


Fig. 18.

Let AB represent the rigid bar one metre long. At A, B, and C are hung 4, 6, 10 grams respectively. It is required to calculate their moments about O when OB $= 4\frac{1}{2}$ decimetres.

Moment = units of mass × units of length in perpendicular from point to line of direction of force;

$\therefore$ Moment of 4 grams is represented by $4 \times 5\frac{1}{2} = 22$.
Moment of 6 grams is represented by $6 \times 4\frac{1}{2} = 27$.
Moment of 10 grams is represented by $10 \times \frac{1}{2} = 5$.

Example 8.—With the same bar compare the moments about the centre of two forces of 10 grams and $\frac{10\sqrt{3}}{\sqrt{2}}$ grams at ends of bar, and respectively making angles of 60° and 45° with the bar.

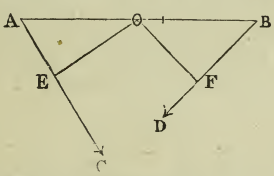


Fig. 19.

Let AB represent the rigid bar one metre long. At A and B act forces represented by 10 grams and $\frac{10\sqrt{3}}{\sqrt{2}}$ grams respectively along the lines AC, BD, making respectively angles of 60° and 45° with AB. It is required to compare the moments of these two forces about O, the middle point of AB.

Moment = units of mass × units of length in *perpendicular* from fixed point.

From O draw lines OE, OF, perpendicular to the directions of AC and BD. Consider $\triangle$ OAE. $\angle$ at E $= 90^\circ$, $\angle$ at A $= 60^\circ$; $\therefore$ $\angle$ at O $= 30^\circ$; $\therefore$ sides OA, AE, EO are as $a, \frac{a}{2}, \frac{a\sqrt{3}}{2}$. But

OA = 5 decimetres; ∴ EO or the perpendicular distance from O to direction of AC $= \frac{5\sqrt{3}}{2}$ decimetres.

Consider △OBF. ∠ at F = 90°, ∠ at B = 45°; ∴ ∠ at O = 45°; ∴ sides OB, BF, FO are as $a, \frac{a\sqrt{2}}{2}, \frac{a\sqrt{2}}{2}$. But OB = 5 decimetres; ∴ OF, or the perpendicular distance from O to direction of BD, $= \frac{5\sqrt{2}}{2}$ decimetres.

The moment of the force of 10 grams $= 10 \times \frac{5\sqrt{3}}{2} = 25\sqrt{3}$.

The moment of the force of $\frac{10\sqrt{3}}{\sqrt{2}}$ grams $= \frac{10\sqrt{3}}{\sqrt{2}} \times \frac{5\sqrt{2}}{2} = 25\sqrt{3}$.

The moments are therefore equal. In the next chapter the two great principles of moments, by the aid whereof the position of the resultant of any number of parallel forces can be found, will be considered.

Keynotes for the third, fourth, and fifth chapters.—(1) Enunciation of triangle of forces. (2) Its proof. (3) To find resultant of three forces not in equilibrium. (4) Enunciation of polygon of forces. (5) To find resultant of many forces not in equilibrium. (6) The magnitude of the resultant of parallel forces. (7) Definition of moment. (8) Its measurement.

EXERCISES.

1. Upon a horizontal bar running due east and west, act a force represented by 10 grams pulling north, a force represented by 15 grams pushing north, two of 8 and 9 grams respectively pulling and pushing south. What force will keep the bar at rest?

2. A bar 8 metres long is fixed at a point midway between its centre and one end. Four weights of 2, 4, 8, 16 grams are attached to it; the two lightest to the ends of the bar, the 8 grams midway between the 2 and the fixed point, the 16 grams midway between the 4 and the fixed point. Draw a diagram to show the arrangement, and compare the moments of the forces.

3. At angles of 120° and 30° to a rigid bar, forces of 12 and 8 grams act. Compare their moments about the middle point of the bar.

CHAPTER VI.

THE TWO PRINCIPLES OF MOMENTS.

(2) Principles of Moments.

THE consideration of parallel forces and of the magnitude of their resultant and its position, has led us to the consideration of moments. Moment is the tendency of a force to turn a body about a fixed point. It is represented numerically by the product of the number of units of force and the number of units of length in the line drawn from the fixed point at right angles to the direction of the force. We pass to the investigation of two great principles in respect to moments, upon the full comprehension whereof depends the solution of all questions upon moments and upon centre of gravity, and of many questions upon machines.

(*a*) *The algebraical sum of the moments of any number of forces about any point in the direction of their resultant is zero.* Suppose a number of forces are acting upon a body, and their resultant is known as to both magnitude and position. All the forces, say, to the left of the resultant, tend to turn the body round in a particular direction. Calculate the several moments of those forces about a point in the direction of their resultant. Regard all these moments as positive. Sum them. Now consider all the forces to the right of the resultant. They tend to turn the body round in a direction contrary to that in which the first set of forces tend to move the body. Calculate the several moments of this right hand set of forces, about that point in the direction of their resultant previously used. Regard all these moments as negative. Sum them. It will be found that their sum is equal to that of the moments of the forces to the left; and as these sums are of opposite sign, the algebraical sum of the moments of the forces about a point in the direction of their resultant is zero.

We proceed to prove this theorem in respect to two parallel forces (i.) acting in the same direction, (ii.) acting in opposite directions, and will show at the same time that the resultant of the forces is equal to their sum.

(i.) Let P and Q represent two forces acting parallel to each other and in the same direction. Along the direction of P, mark off BA to represent P. Along the direction of Q, mark off DC to represent Q. At B and D introduce two equal and opposite forces, represented by BE and DF respectively. Combine the two forces represented by BA and BE, and find a line representing

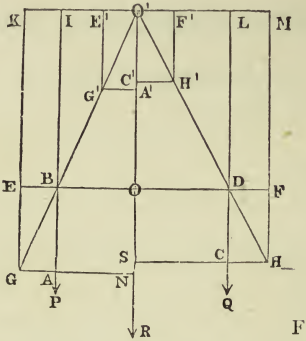


Fig. 20.

their resultant, BG. Combine the two forces represented by DC and DF, and find a line representing their resultant, DH. Produce GB and HD until they meet. Let the point of meeting be called O′. Along O′B mark off O′G′ = BG. Decompose force represented by O′G′ into a vertical force represented by O′A′, and a horizontal represented by O′E′. Along O′D mark off O′H′ = DH. Decompose force represented by O′H′ into a vertical force represented by O′C′, and a horizontal represented by O′F′. At O′ ∴ act two equal and opposite forces, represented by O′E′, O′F′. These are in equilibrium, and can be eliminated. Thus are left acting at O′ two vertical forces, represented by O′A′, O′C′ = P and Q respectively; ∴ resultant of P and Q is a force = their sum acting at O immediately beneath O′.

Through O′ draw a line parallel to BD. Extend lines GE, AB, CD, HF until they meet this line. Extend GA, HC until they meet the vertical line drawn through O′O. The moment of the force P about O, a point in the direction of the resultant of P and Q, equals the units of force in P × the units of length in the line drawn from O perpendicular to the direction of P, = AB × BO = rectangle AO. The moment of the force Q about O, a point in the direction of the resultant of P and Q, equals the units of force in Q × the units of length in the line drawn from O perpendicular to the direction of Q, = CD × DO = rectangle CO. But AO and KB are the complements of the parallelograms EA and OI about the diameter GO′ of the parallelogram KN, and are equal. CO and DM are the complements of the parallelograms FC and LO about the diameter HO′ of the parallelogram MS, and are equal. But KB and MD are parallelograms on equal bases BE and FD, and between the same parallels, and are therefore equal. Now, as AO = KB, CO = DM, and KB = DM; ∴ AO = CO, or the rectangles representing the moments of P and Q about a point O in the direction of the resultant of P and Q are equal numerically. The moments are also of opposite sign, as P tends to turn the bar round O as fixed point in one direction, and Q tends to turn the bar round O as fixed point in the opposite direction.

(ii.)

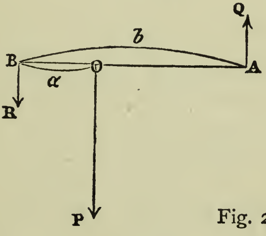


Fig. 21.

Let P and Q represent two forces parallel to each other, but acting in opposite directions. Their resultant R = their algebraical sum or their arithmetical difference, *i.e.* P − Q. Let a = perpendicular distance of B, the point where R acts from the direction of P, and let b = perpendicular distance of B from the direction of Q. When three forces are in equilibrium, any one of them is equal and opposite to the resultant of the other two; ∴ P is equal and opposite to resultant of R or (P − Q) and Q. Hence the moments of R or (P − Q) and Q about a point in the direction of P, are equal numerically; ∴ Ra or $(P - Q)a = Q(b - a)$; ∴ $Pa - Qa = Qb - Qa$; ∴ $Pa = Qb$, or the moment of P about a point B in the resultant of P and Q = the moment of Q about a point B in the resultant of P and Q.

The proofs of the above in respect to two forces can be extended to any number of forces, as all those acting in one direction can be represented as one force = their sum, and all those acting in the opposite direction can be represented as one force = their sum.

Example 9—Three parallel forces, represented respectively by

8, 6, 4 grams, act on a rigid bar. The 8 grams act at one end, the 6 grams at a distance of 4 decimetres from the same end. The resultant of all the forces acts at a point 3 decimetres from the 8 grams. Find the length of the bar if the 4 grams acts at the end opposite to the 8 grams.

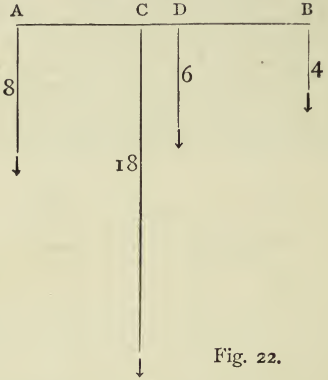


Fig. 22.

Let AB represent the rigid bar. At the end A acts the force represented by 8 grams. The resultant acts at C, a point 3 decimetres from A. The force represented by 6 grams acts at a point D, 4 decimetres from A. The force represented by 4 grams acts at the end B, opposite to A. It is required to find the length of AB. Let BC = x.

The algebraical sum of the moments of any number of forces about a point in the direction of their resultant is zero; $\therefore$ algebraical sum of moments of 8, 6, 4 about C = 0. Moment of force represented by 8 grams = 8 × units of length in AC = $8 \times 3 = 24$. Moment of force represented by 6 grams = 6 × units of length in CD = $6 \times 1 = 6$. Moment of force represented by 4 grams = 4 × units of length in BC or $x = 4x$. The moment of the 8-gram force tends to turn the bar in one direction about C. The moments of the other two forces, 6 grams and 4 grams, tend to turn the bar in the opposite direction about C.

$$24 - 6 - 4x = 0.$$
$$18 = 4x,$$
$$x = 4\tfrac{1}{2}.$$

Whole length of AB = AC + CB = $3 + 4\frac{1}{2} = 7\frac{1}{2}$ decimetres.

(*b*) *The algebraical sum of the moments of any number of forces about any point in their plane = the moment of the resultant of those forces about the same point.*

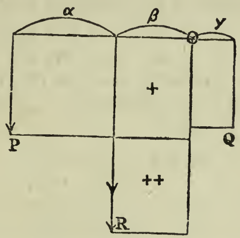


Fig. 23.

Let O be the point in the plane of P and Q the forces and $R (= P + Q)$ their resultant. Moment of $P = P(\alpha + \beta)$, of $Q = Q\gamma$, of $R = R\beta$. The parallelogram marked + is common to those representing the moments of P and R about O. We have left $P\alpha - Q\gamma$ to be shown equal to parallelogram marked + +. This last $= Q\beta$. But $P\alpha = Q(\gamma + \beta)$ as moments of forces about any point in direction of their resultant are numerically equal; $\therefore\ P\alpha - Q\gamma = Q\beta$. And this is that which we desired to show.

Example 10.—The same three forces act upon the same bar as in the last example. Determine the position of their resultant.

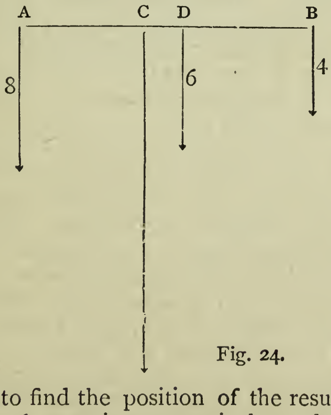


Fig. 24.

It is required to find the position of the resultant of the three parallel forces, 8, 6, 4, acting respectively at the points A, D, B. The algebraical sum of the moments of any number of forces about any point in their plane is equal to the moment of the resultant about that same point.

Consider the point B; calculate moments about B.

$$\begin{aligned} \text{Moment of 8 grams} &= 8 \times 7\tfrac{1}{2} = 60, \\ \text{,, \quad 6 \quad ,,} &= 6 \times 3\tfrac{1}{2} = 21, \\ \text{,, \quad 4 \quad ,,} &= 4 \times 0 = 0. \\ \text{Resultant} &= 8 + 6 + 4 = 18 \text{ grams.} \end{aligned}$$

Let x represent its perpendicular distance from B. Then moment of resultant $= 18x$;

$$\begin{aligned} \therefore\ 60 + 21 + 0 &= 18x, \\ 81 &= 18x, \\ 9 &= 2x, \\ x &= 4\tfrac{1}{2}. \end{aligned}$$

Resultant acts at point C, $4\frac{1}{2}$ decim. from B.

EXERCISES.

1. Find the magnitude and position of the resultant of four weights of 1, 2, 3, 4 grams, acting at distances of 4, 3, 2, 1 decimetres from one end of a bar 1 metre long.

2. Weights of 20, 60, 100 grams are suspended from the extremities and middle point of a bar 6 decimetres long, the 60 grams in the middle. What weight must be placed at a point midway between the points of suspension of the 20 and 60 grams for the bar to balance about the centre point?

3. The resultant of three parallel forces equals 34 grams. If one force equals a decagram, and acts at a distance equal to a metre from resultant, what is the value of the two other forces, supposing they act at distances of 3 and 5 decimetres respectively upon the opposite side of the resultant to the first?

4. Two parallel forces act in opposite directions. One is three-fourths of the magnitude of the other. The distance of the lesser from the resultant equals 4 decimetres. If the resultant equal 10 grams, find the forces and the distance between them.

5. On a horizontal bar 1·5 metres long, fixed at one end, are hung in succession weights of 5, 4, 3, 2, 1 grams. The distance of the 5 grams from the fixed end equals 1 decimetre. The successive distances between the pairs of weights increase in arithmetical progression, taking the distance between the 5 grams and the fixed end as the first term. Find the lengths of these successive distances when a force equal to 70 grams, acting at the

point of suspension of the 5 grams, but in opposite direction to the forces, maintains equilibrium.

6. Four parallel forces, acting in same direction, have resultant equal to 32 grams. The distances between the pairs of weights are 5, 4, 3 decimetres. If the second, reckoning from the left, equal 8 grams, the two extreme ones are equal, and the fourth equals twice either of the last, find value of forces and position of resultant.

7. A bar is fixed at one end. From the remote end hangs a weight of 4 grams. A force of $5\frac{1}{3}$ grams acts upwards at a point 2 decimetres from the 4 grams, 10 grams depend from the middle of the bar, and a force $= 40$ grams acts upwards at a point 1 decimetre from the fixed end. The bar remains horizontal. Find its length.

8. What force acting at an angle of 45° with a horizontal bar 1 foot long will balance a vertical force of $10\sqrt{2}$ pounds, the two forces acting at the extremities of the bar, and the bar balancing about its middle point?

For keynote (5), page 32, a student writes:—'Find resultant of two forces; combine this resultant with the third force; combine this second resultant with the fourth force, and so on.' It would have been better to have written:—'Find resultant of two forces; then of two others, and so on, until you have resultants of pairs of forces; combine these resultants in pairs, and so on, until one final resultant is obtained.'

CHAPTER VII.

B. CENTRE OF GRAVITY.

(1) Definition.

From the study of moments we pass to that of the centre of gravity. Let the student hold on the palm of his hand the book now under study. A pressure is felt upon the hand. That pressure is said to be due to the weight of the book. The cause of the weight of the book is, of course, the attraction exercised upon the book by the mass of the earth below. All other bodies in the universe are dragging at the book and trying to draw it towards them, and this universal attraction of matter for matter working through enormous distances, and by no means requiring that the bodies attracting one another should be near together, is called *gravitation*. In its power of acting through great distances, gravitation differs from the three other forms of attraction met with in the study of Physics. These are cohesion, adhesion, chemical attraction. Now *cohesion*, illustrated by the attraction, say, of the atoms of the paper of the book one for the other, is a form of attraction between atoms near together and alike. *Adhesion*, illustrated by the attraction between the atoms of the printer's ink and the atoms of the paper to which it adheres, is the attraction between atoms near together and unlike. *Chemical action*, illustrated by the attraction between copper and dilute nitric acid, is the attraction between atoms near together and unlike; but when chemical attraction occurs, a change of properties, unpredicable until the experiment has been made, also results. Thus when dilute nitric acid is poured upon copper, a gas at first colourless, but turning red-brown upon contact with air, and a liquid of beautiful bluish green are formed; so that from the chemical attraction between a solid metal and a yellowish acid liquid, and from their action one upon another, result a colourless gas that, joining with colourless oxygen from the air, becomes red-brown, and a bluish-green liquid not acid at all. There is great change of properties. Summing up these four great physical attractions in a tabular arrangement, and leaving out of our consideration electrical and magnetic attractions as not of the like universal nature to these, we have the following arrangement:—

Attraction between atoms at a distance.			Gravitation.	Earth to sun.
Attraction between atoms near together.	Atoms alike.		Cohesion.	Wood to wood.
	Atoms unlike.	No change of properties.	Adhesion.	Ink to paper.
		Change of properties.	Chemical attraction.	Copper to nitric acid.

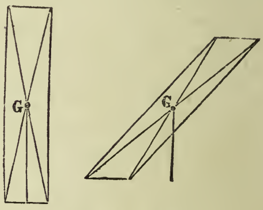
Gravitation, therefore, with which alone we have now to do, acts between atoms at any distance however great from each other. But the greater the distance between the bodies attracting one another, the less is the force of the attraction. Hence, returning to our book illustration, when the book or any other solid body is taken in the hand, the attraction of the huge earth mass so close at hand dominates all other attraction of planets, suns, and stars; and you, feeling the pressure on your hand due to this dominating attraction, speak of the *weight* of the book. Hence arises the common definition of gravitation as the attraction of bodies to the earth. This attraction of bodies to the earth is only one special instance of gravitation.

The book you hold in your hand is made up of atoms whose number is inconceivably great. Each one of this infinite number of atoms is attracted towards the mass of the earth. Assuming that the geometrical centre of the earth is the centre of that mass, we may imagine a line drawn from every one of the countless atoms of the book to the centre of the earth, and representing the line of direction of the force tending to move that atom towards the earth. There will thus be quite innumerable lines representing forces passing from a small book to a point that is at a distance of about 4000 miles. So innumerable are these lines, so immense their length, so small the area of the book whence they are all in imagination drawn, that we may look upon them as parallel. Hence the forces acting upon a body in consequence of its gravitation towards the earth are practically parallel forces. To them, therefore, the principles of moments will apply.

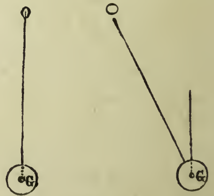
Their resultant will equal the sum of these individual forces, and this is called the weight of the body. Again, the algebraical sum of the moments of any number of forces about a point in the direction of their resultant is zero. Place the book flat upon the hand. Now there is a vertical plane running through the middle of the book, about any point wherein the sum of the moments of all these parallel forces due to the earth's attraction will be zero. Place the book on its edge or balance it on its back. There is now another vertical plane running through the middle of the book as it now stands, about any point wherein the sum of the moments of all the parallel forces due to the earth's gravitation will be zero. These two planes intersect in a particular line. Suspend the book by one of its corners; yet another vertical plane, having like properties to the other two, is found. These three planes intersect in a point, and that point is the centre of gravity.

The centre of gravity of any body or system of bodies is the point through which in any position of the body or system of bodies passes the resultant of all the forces acting upon that body in consequence of the attraction of gravitation between the body or system of bodies and the earth.

A practical result of great moment in relation to the centre of gravity next calls for notice. If the vertical line through the centre of gravity passes through the point or area of support, the body if a solid is supported. If the vertical line through the centre of gravity does not pass through the point or area of support, the body falls. Hence, in rough phrase, if the centre of gravity is supported, the whole body is supported. Thus the cylinder (1), Fig. 25, would stand firmly, as the vertical line drawn through its centre of gravity G falls within the base. But (2) would fall, as the vertical line drawn through its centre of gravity does not fall within the base. Again, the ball suspended by a cord in (3) rests still, for the vertical line through its centre of gravity passes through the point of suspension O. But the ball in (4) swings down, because the vertical line through its centre of gravity does not pass through the point of suspension, and this ball will not be at rest until after friction and the resistance of the air have long acted upon it as it swings, it comes into the same position as in (3), and the vertical line through its centre of gravity passes through the point of support O.



(1) Fig. 25. (2)



(3) Fig. 26. (4)

(2) Conditions of Equilibrium,

The three conditions of equilibrium of bodies at rest, relatively to the earth, under the action of the earth's attraction, depend upon the position of the centre of gravity in respect to the point or basis of support. These three conditions are—stable equilibrium, unstable equilibrium, neutral equilibrium.

(*a*) *Stable equilibrium.*—When any lateral displacement of a body raises its centre of gravity, this form of equilibrium obtains. Thus a ball A suspended by a string from a point O, is in stable equilibrium. Any lateral displacement of it, as when A is removed to the position of A′, raises the centre of gravity. Further, after such lateral displacement, the body always tends to return to its original condition of stable equilibrium. A′ when released, will swing back towards the position of A, and finally remain therein.

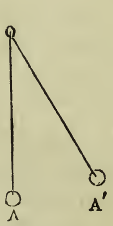


Fig. 27.

(*b*) *Unstable equilibrium.*—When any lateral displacement of a body lowers its centre of gravity, this form of equilibrium obtains. Thus, a stick A, supported on a point O, is in unstable equilibrium. Any lateral displacement of it, as when A is removed to the position A′, lowers the centre of gravity. Further, after such lateral displacement, the body does not tend to return to its original condition of unstable equilibrium, but to fall until it is below the point of support as in case (*a*), or level with the basis of support, and is in the condition of neutral equilibrium next to be studied.

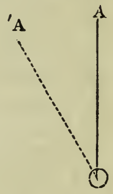


Fig. 28.

(*c*) *Neutral equilibrium.*—When any lateral displacement of a body neither raises nor lowers its centre of gravity, but keeps it at the same level as before, this form of equilibrium obtains. Thus a ball A, supported on a horizontal plane, is in neutral equilibrium. Any lateral displacement of it, as when A is removed to the position A′, leaves the centre of gravity at exactly the same height as before. Further, after such

Fig. 29.

lateral displacement, the body does not tend to return to its former condition nor to go further away from its former condition. It remains exactly where it is left, and is still in a state of neutral equilibrium.

(3) To find the Centre of Gravity.

(*a*) *Experimentally.*—The centre of gravity of any body, no matter how irregular its shape, may be found thus. Suspend the body by a string attached above to a fixed point, below to any point in the body. Wait until the arrangement is quite at rest. Two forces are now in equilibrium, viz. the weight of the body and the tension in the string. But when two forces are in equilibrium, they must be equal and opposite. The tension in the string and the weight of the body are clearly equal; they are also opposite. Now the line of the string represents the direction of the tension of the string. Therefore the weight of the body being opposite to the tension of the string in direction, must act in the same straight line as the direction of the string; therefore the centre of gravity of the body must be in the same straight line as the direction of the string. Produce then the line of direction of the string, and the centre of gravity of the body will be somewhere in the line thus drawn. Next suspend the body by a string attached above to a fixed point and below to a second point in the body. By reasoning similar to that just given, the line of direction of the string in this second case must pass through the centre of gravity of the body. As the centre of gravity must be somewhere in this new line, and also somewhere in the former line that was drawn, it must be in the point where these two lines intersect.

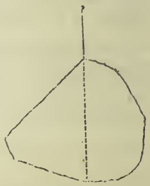


Fig. 30.

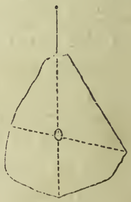


Fig. 31.

(*b*) *Geometrically.*—The centre of gravity of a uniform line is at its point of bisection. That of a circle is at the geometrical centre. That of a parallelogram of any form is at the point of intersection of its diameters. The position of the centre of gravity of a triangle is of great importance, as upon the knowledge thereof depends the determination of the position of the centre of gravity of many other figures.

To find the centre of gravity of a uniform triangle.—Let EDI represent a triangle. It is required to find the position of the centre of gravity of that triangle. Bisect the side EI. Let the point of bisection be T. Join T with the opposite angle D. The whole triangle may be considered to be made up of an infinite number of lines parallel to EI. Each of these will be bisected by the line TD, and therefore somewhere in the line TD must be the centre of gravity of the whole triangle.

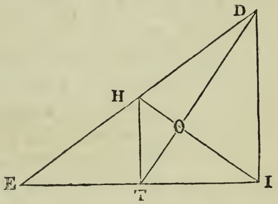


Fig. 32.

Again, bisect the side ED. Let H be the point of bisection. Join H with the opposite angle I. The whole triangle may be considered to be made up of an infinite number of lines parallel to ED. Each of these will be bisected by the line HI; therefore somewhere in the line HI must be the centre of gravity of the whole triangle. But as it must also be in the line TD, the centre of gravity can only be at the point of intersection of the lines TD and HI, *i.e.* at the point O.

To determine the exact position of this point O, join T and H, the two points of bisection of the sides EI and ED. By Euclid (Book VI. Prop. 2) the line HT is parallel to DI the third side, and it can be shown to be equal to one-half DI. Consider the two triangles HOT, DOI. $\angle HOT = \angle DOI$ (vertically opposite angles). Also the alternate angles OHT and OID are equal, and the alternate angles HTO, IDO are equal; therefore the triangles HOT, DOI are equiangular, and their sides must be proportional.

But $HT = \frac{DI}{2}$; $\therefore$ $TO = \frac{OD}{2}$ and $TO = \frac{TD}{3}$. Also $HO = \frac{OI}{2} = \frac{HI}{3}$. Hence the centre of gravity of any uniform triangle is at a point in the line joining the point of bisection of one of the sides with the opposite angle, and at a distance of one-third of that line from the point of bisection, or two-thirds of that line from the opposite angle.

The centre of gravity of figures other than circles, parallelograms, or triangles, can, as a rule, be found by dividing them into triangles, and applying the proof just given. Thus, if it were required to find the centre of gravity of the irregular quadrilateral ZYXW. Divide it into two triangles by joining W and Y. Bisect the line WY; let the point of bisection be A. Join A with Z and with X. Find centre of gravity of △YZW; let it be G. Find centre of gravity of △YXW; let it be G′. Join G and G′. Centre of gravity of whole figure must be somewhere in the line GG′.

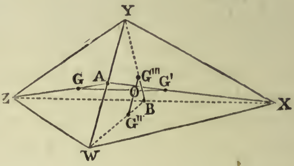


Fig. 33.

Again, divide the figure into two triangles by joining Z and X. Bisect the line ZX; let the point of bisection be B. Join B with W and with Y. Find centre of gravity of △ZWX; let it be G″. Find centre of gravity of △ZYX; let it be G‴. Join G″ and G‴. Centre of gravity of whole figure must be somewhere in the line G″G‴. And as it is also in the line GG′, it can only be in the point of intersection O of these two lines GG′ and G″G‴.

Figures of more than four sides can also be dealt with in a similar way if they are regular. Thus, the centre of gravity of half a regular hexagon can be found, after the method shown in the accompanying diagram.

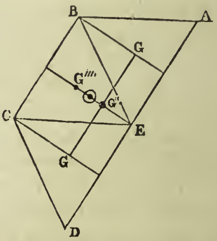


Fig. 34.

Centre of gravity of △ABE is at G.
,, ,, ,, ,, DCE is at G′.
,, ,, ,, ,, both triangles is at G″.
,, ,, ,, ,, △CBE is at G‴.
,, ,, ,, ,, whole figure is at O, one-third up line from G″ to G‴.

(*c*) *Arithmetically.*—The principles of moments come into play. Those two great principles are—(i.) that the algebraical sum of the moments of any number of parallel forces about any point in the direction of their resultant is zero; (ii.) that the moment of the resultant of any number of parallel forces about any point in their plane equals the algebraical sum of the moments of the forces about the same point.

Suppose it is required to find the position of the centre of gravity of four weights weighing respectively 1, 2, 4, 8 grams, placed on a horizontal rod 8 decimetres long, the first and last at the two ends, and the other two at equal distances from the ends and the centre of the bar.

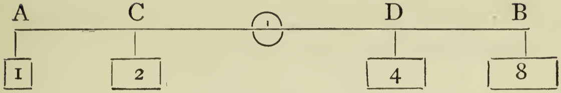


Fig. 35.

Let AB represent the horizontal rod, 8 decimetres long. At A hangs a weight of 1 gram; at B, one of 8 grams; at C, halfway between A and O the centre of the bar, one of 2 grams; at D, midway between B and O, the centre of the bar, one of 4 grams. It is required to find the position of the centre of gravity of the balls, the weight of the rod not being taken into consideration.

Apply principle (ii.) given above. Calculate moments about point B. Any point may be selected, but it is best to choose a point where one of the forces is acting, and, if possible, one of the extreme points. Let x = perpendicular distance of point of application of the resultant of these four forces from B. The numerical value of the resultant $= 1+2+4+8=15$.

By principle (ii.),

$$15x=(1\times 8)+(2\times 6)+(4\times 2)+(8\times 0)=8+12+8+0,$$

$$15x=28,$$

$x=28\div 15=1\frac{13}{15}$ decimetres from B, or a little to the right of D.

Example 11.—An equilateral triangle weighing 4 grams has weights of 2 grams attached to two of its corners. Find the centre of gravity of the whole if one side of the triangle is $16\sqrt{3}$ centimetres long.

BRA is a triangle weighing 4 grams. At B and A are suspended weights of 2 grams. The side of the triangle is $16\sqrt{3}$ centimetres long.

To find centre of gravity of triangle, bisect BA. Let D be the point of bisection. Join DR. Find L a point $\frac{1}{3}$ up DR. This is the centre of gravity of the $\triangle$. Consider $\triangle$BRD. $\angle$ at D $=90°$, $\angle$ at B $=60°$, $\angle$ at R $=30°$;
$\therefore$ sides BR, BD, RD, are as

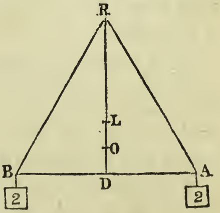


Fig. 36.

$$a, \frac{a}{2}, \frac{a\sqrt{3}}{2}.$$

But $BR = 16\sqrt{3}$ centimetres;

$$\therefore\ a \text{ represents } 16\sqrt{3} \text{ centimetres},$$

$$1 \text{ represents } \frac{16\sqrt{3}}{a},$$

$$\frac{a\sqrt{3}}{2} \text{ represents } \frac{8\ \cancel{16}\sqrt{3}}{\cancel{a}} \times \frac{\cancel{a}\sqrt{3}}{\cancel{2}} = 8 \times 3 = 24 \text{ centimetres};$$

$$\therefore\ RD = 24 \text{ centimetres};$$

$$\therefore\ DL = 8 \text{ centimetres}.$$

To find centre of gravity of the two weights of 2 grams. D, the middle of BA, is the position of this centre of gravity.

To find that of whole system. At L acts a force represented by 4 grams; at D acts a force represented by 4 grams. The centre of gravity of the whole system is at the point of bisection of LD or at O. But LD = 8 centimetres; ∴ O is 4 centimetres from D.

Example 12.—At the middle and either end of a rod 1 metre 8 decimetres in length are suspended weights of 3, 6, and 9 grammes respectively. If the weight of the rod is 3 grammes, where will be the C. G. of the system?

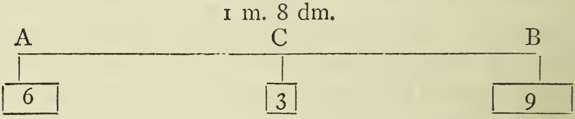


Fig. 37.

Let AB be a rod 18 decimetres long, and C the middle point; let a weight of 3 grammes be at C, a weight of 6 grammes at A, and a weight of 9 grammes at B; let the weight of AB be also 3 grammes.

It is required to find the C. G. of the whole system.

Apply the second principle of moments. Take the point B, the point of application of the heaviest weight; then the algebraical sum of the moments of the forces about this point will equal the moment of the resultant about the same point.

Four forces are to be considered:—the weight of 6 grammes acting at A; the weight of 3 grammes acting at C; the weight of 9 grammes acting at B; the weight of the rod, taken as concentrated at its middle point, therefore = 3 grammes acting at C.

The moment of (6) about B $= 6 \times 18 = 108$
,, ,, ,, (3) ,, $= 3 \times 9 = 27$
,, ,, ,, (9) ,, $= 9 \times 0 = 0$
,, ,, ,, (3) ,, $= 3 \times 9 = 27$
162

The moment of the resultant is the sum of the weights $(6+3+9+3=21) \times$ perpendicular distance x from the point B;

$$\therefore\ 21x = 162,$$

$$\text{and } x = \frac{\overset{54}{\cancel{162}}}{\underset{7}{\cancel{21}}} = 7\tfrac{5}{7};$$

therefore the C. G. is at a distance of $7\frac{5}{7}$ decimetres from B.

Example 13.—It is required to find the centre of gravity of a square plate of uniform thickness, one-half of which is made of copper, and the other half of tin, the relative weights of the two metals being 8·788 : 7·291, and their line of juncture forming the diagonal of a square whose length is 24 inches.

Let APBQ in Fig. 38 represent the square plate, APB the copper triangle, and AQB the tin triangle.

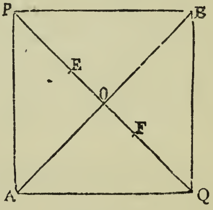


Fig. 38.

Let the unit of weight be taken so that the △APB weighs 8·788 units; then since △ABQ = △APB, and since the weight of tin : the weight of copper : : 7·291 : 8·788;

∴ the △ABQ weighs 7·291 units;

∴ a resultant force of 8·788 units acts through the centre of gravity of the △APB, and a resultant force of 7·291 units acts through the centre of gravity of the △ABQ.

Find the centre of gravity of each triangle, and let E be the point wherewith the centre of gravity of the △APB coincides, and F be the point wherewith the centre of gravity of the △ABQ coincides.

Join EF, and let EF cut the line of juncture AB in O.

Then it is proved by the process of finding the centre of gravity of a △ (see above), that if EF is prolonged both ways, it will pass through the points P and Q, and that

$$\mathrm{OF} = \tfrac{1}{3}\mathrm{OQ},$$
$$\text{and } \mathrm{OE} = \tfrac{1}{3}\mathrm{OP} = \tfrac{1}{3}\mathrm{OQ} = \mathrm{OF}.$$

In the line EF we have a weight of 8·788 units at the point E, and a weight of 7·291 units at the point F; and the resultant of these two forces will by the definition of centre of gravity coincide with the centre of gravity of the whole square.

Let x be the number of inches that the required centre of gravity is from O. Since the weight at E>weight at F, this distance must be measured to the left of O in the diagram;

∴ by the principle of moments,

$$8{\cdot}788 \times (\text{number of inches in OE} - x)$$
$$= 7{\cdot}291 \times (\text{number of inches in OF} + x);$$
$$\therefore\ 8{\cdot}788\ (\tfrac{24}{2} \times \tfrac{1}{3} - x) = 7{\cdot}291\ (\tfrac{24}{2} \times \tfrac{1}{3} + x);$$
$$\therefore\ 8{\cdot}788\ (4 - x) = 7{\cdot}291\ (4 + x);$$
$$\therefore\ x(7{\cdot}291 + 8{\cdot}788) = 4(8{\cdot}788 - 7{\cdot}291);$$
$$\therefore\ 16{\cdot}079 \times x = 5{\cdot}988;$$
$$\therefore\ x = \frac{5{\cdot}988}{16{\cdot}079} = {\cdot}37 \text{ approximately.}$$

Wherefore the centre of gravity of the plate coincides with a point in the line PQ (which is the diagonal line of juncture), situated between the centre O of the square and the apex P of the copper △, and approximately ·37 of an inch from the point O.

Example 14.—A solid rod of square cross section, and length 3 metres, is made up of 2 rods of unlike materials, placed end to end. The piece to the right is ½ metre long, and the centre of gravity of the whole is 2 metres from the left extremity. What are the relative weights of the component bars?

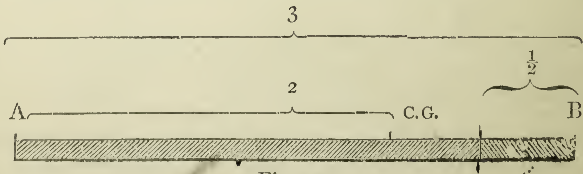


Fig. 39.

The length of the left-hand rod $= 3 - \tfrac{1}{2} = 2\tfrac{1}{2}$ metres.

The distance of the centre of gravity from the point of juncture between the two rods $= 2\tfrac{1}{2} - 2 = \tfrac{1}{2}$ metre.

Let 1 metre of the material of the right-hand rod weigh y units of weight.

Let 1 metre of the material of the left-hand rod weigh x units of weight;

$$\therefore\ 2 \text{ metres of the latter weigh } 2x,$$
$$\frac{1}{2}\quad ,,\quad ,,\quad ,,\quad ,,\quad \frac{1}{2}x,$$
$$\frac{1}{2}\quad ,,\quad ,,\quad \text{former}\quad ,,\quad \frac{1}{2}y;$$

∴ the centre of gravity of the whole rod is the point through which the resultant passes of a force represented by $\frac{y}{2}$, acting at a distance of $\left(\frac{1}{4}+\frac{1}{2}\right)$ of a metre to the right of that centre, a force of $\frac{x}{2}$, at a distance of $\frac{1}{4}$ metre, acting in the same direction, and a force of $2x$, acting at a distance of 1 metre to the left of the centre.

$$\text{Wherefore } \frac{3}{4}\times\frac{y}{2}+\frac{1}{4}\times\frac{x}{2}=2x\times 1;$$
$$\therefore\ 3y+x=16x;$$
$$\therefore\ 3y=15x;$$
$$\therefore\ \frac{y}{x}=\frac{15}{3}\text{ cm.}=\frac{5}{1}\text{ cm.};$$

∴ the weight of unit length of the shorter rod : the weight of unit length of the longer rod : : 5 : 1.

EXERCISES.

1. A uniform rod is composed of 3 parts placed end to end. The relative weights of the materials whereof the parts are made are $m : n : p$ (beginning at the right hand), and their relative lengths are $r : s : t$. Find the centre of gravity of the whole, having given that it lies between the points of juncture of the rods.

2. A uniform horizontal triangle weighing 10 grams has a weight hung from one angle. The C. G. of the whole is in the line joining the point with the middle of the opposite side, and $\frac{5}{9}$ of the length of that line from the point. What is the weight?

3. Along a rod 5 decimetres long are ranged at equal distances six weights respectively of 4, 5, 3, 2, 5, 4 grams, the two weights of 4 grams at the extremities of the rod. Find position of the centre of gravity of the whole.

4. A rod weighing 20 grams supports, and whose C. G. is 7 centimetres from its middle point, at a point 2 decimetres from one end, a weight. Three decimetres from the other end is suspended a weight equal twice the former one. What are the weights, if the rod being 1·5 metres long, balances about a point $7\frac{1}{2}$ decimetres from the end near the heavy weight?

5. A bar 2 metres long has an equilateral triangle attached by one side to the bar at one end in such a manner that the side of the triangle exactly coincides with the terminal 6 decimetres of the bar. If the triangle and the bar have the same weight, determine the position of the centre of gravity of the system.

6. To the angles of a rectangle 4 decimetres long by 6 broad, are affixed weights taken in order of 2, 6, 4, 4 grams, the two equal weights at the extremities of one of the short sides. Find the distance of the centre of gravity of the four weights from the centre of the rectangle.

CHAPTER VIII.

C. MACHINES.

The principles of moments being mastered, it is now possible to turn to the investigation of Machines. All the complex machinery encountered in a cotton mill, or in any other busy haunt of labour, carefully examined, resolves itself into some half-a-dozen simple structures. The whole of the machinery in use to-day throughout the world, consists of a few simple machines, as all the words of our language are made up of one, two, or more of some twenty-six letters. It is now necessary to study these simple machines.

(1) EXPLANATION OF TERMS.

A machine, say *e.g.* a pulley, is an arrangement whereby man's labour or man's time is saved. A more accurate definition is, that a machine is an arrangement whereby a force exerting power in one direction may overcome a resistance not immediately opposed to it. Thus, in drawing up a window blind, a servant pulling *downwards* overcomes the resistance or weight of the blind, also acting *downwards.* This is effected through aid of a simple machine or pulley, fixed at the upper part of the room. Certain terms are in general use in relation to machines. The original force employed to do work is called the Power, and will be represented by the letter P. That which is to be overcome is called the Resistance or Weight, and will be represented by the letter W. In general, the contrivance called a machine is of such nature that a lesser Power overcomes a greater Weight, as when a man pushes a heavy barrel up an inclined plane with a force of less magnitude than the weight of the barrel.

The ratio of the Weight to the Power in any machine or system of machines is known as the mechanical advantage of the machine or set of machines, or as the Modulus of the machine or set of machines. Thus, if in different machines Weights respectively of 10, 8, 13, 1, 2, exactly balance Weights respectively of 1, 2, 7, 1, 4, the respective mechanical advantages are $\frac{10}{1}$, $\frac{8}{2}$, $\frac{13}{7}$, $\frac{1}{1}$, $\frac{2}{4}$, or 10, 4,

$\frac{13}{7}$, 1, $\frac{1}{2}$. In the last case the mechanical advantage is fractional, or in ordinary phrase, the machine works at a disadvantage.

Another general principle of great importance is, that where there is advantage in the direction of force, there is disadvantage in another direction. If a lesser force overcomes a greater one, the former will have to move through a greater distance than the latter. Thus, in the case given above of a man pushing a barrel up an inclined plane, the man's hand moves over a distance equal to the base of the plane, whilst the barrel is raised a distance equal to the height of the plane. Clearly there can be no creation of force, and universally in machines it is found that *the product of the Power and the distance through which it moves is equal to the product of the Weight and the distance through which it moves.* Having therefore defined a machine as an arrangement whereby a force known as the Power can balance or overcome a resistance known as the Weight when the two are not directly opposed one to the other, and having laid down the great principle that the product of the number of units in the Power and the number of units of length in the distance through which it moves = the product of the number of units in the Weight and the number of units of length in the distance through which it moves, we pass to the consideration of the simple machines or mechanical powers out of which all machinery is constructed.

(2) SIMPLE MACHINES.

(*a*) THE LEVER.

(i.) Definition.

A rigid bar moveable about a fixed point. The fixed point is called the fulcrum. Imagine a man poking the fire. His muscular effort is the Power; the resistance of the coal is the Weight; the bar of the grate on which the poker rests is the fulcrum; the poker is a lever. Imagine a man lifting a stone by aid of a crowbar: his muscular effort is the Power; the weight of the stone is the Weight; the ground against which the end of the crowbar rests is the fulcrum; the crowbar is a lever. Imagine a woman working a sewing-machine with a treadle: her muscular effort exerted through her foot is the Power; the resistance of the machine is the Weight; the hinge where the treadle is joined on to the fixed part of the machine is the fulcrum; the treadle of the sewing-machine is a lever. It will be noticed that in the three examples given, the relative positions of Power, Weight, and fulcrum vary. Indeed, the three cases given illustrate the three kinds of lever. The poker is of the first kind, where the fulcrum is in the middle, Fig. 40.

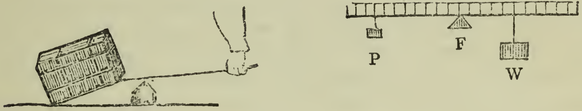


Fig. 40.

The crowbar, as used in the example given, is of the second kind, where the Weight is in the middle, Fig. 41.

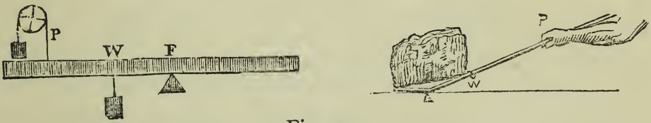


Fig. 41.

The treadle of the sewing-machine is of the third kind, where the Power is in the middle, Fig. 42.

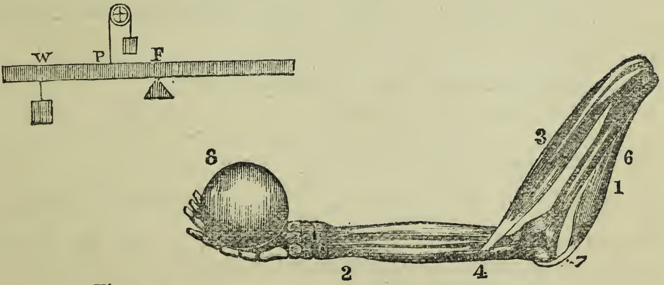


Fig. 42.

(ii.) When the Weight of the Lever is not considered.

The perpendicular distances from the fulcrum or fixed point to the lines of direction of the Power and Weight, are called the arms of the Power and Weight respectively. Thus, in the above diagrams, the line from the point of action of P to F represents the Power's arm, and the line from the point of action of W to F represents the Weight's arm. Hence in the first kind of lever the whole length of the lever = P's arm + W's arm. In the second kind of lever the whole length of the lever = P's arm. In the third kind of lever the whole length of the lever = W's arm. If the Power and Weight act at an angle other than right upon the lever, the *perpendicular* distance between the fulcrum and the lines of their direction must be measured to determine the length of their arms.

One great principle will solve all cases of the lever. It is one of the principles of moments, viz., that the algebraical sum of the moments of any number of forces about a point in the direction of their resultant is zero. In the lever two forces only, as a rule, demand consideration. They are the Power and the Weight. The fulcrum or fixed point is the point in the direction of their resultant when there is equilibrium. The perpendicular distance from the fulcrum to the lines of direction of Power and Weight are the arms of the Power and Weight. Hence, generally, in the lever there is equilibrium when the product of the number of units of force in the Power and of the number of units of length in the Power's arm = the product of the number of units of force in the Weight and the number of units of length in the Weight's arm, or briefly, when P × its arm = W × its arm.

Example 15.—If in a lever of the second kind, the weight is four times the Power, and the arm of the Weight is 3 centimetres, find the distance between the points of application of the Power and Weight.

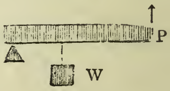


Fig. 43.

The diagram represents a lever of the second order, with the Weight W acting between the Power $P = \frac{W}{4}$ and the fulcrum F, and at a distance of 3 cm. from F. It is required to find the distance between the points of application of P and W.

P × its arm = W × its arm. Let x = length of lever;

$$\therefore \frac{W}{4} \times x = W \times 3; \;\therefore \frac{x}{4} = 3; \;\therefore x = 12.$$

$12 - 3 = 9$ is the required distance.

Example 16.—In testing for the accuracy of a rough balance composed of a horizontal bar AB, 1 metre long, supported at F, it is found that a load of weight 3 grams hung at one extremity, balances another of 2·9 grams hung at the other extremity. In which direction and through what distance should the point of support be shifted?

The balance is a lever of the first order. The point of support is the fulcrum. P and W are represented by the loads 2·9 and 3 respectively. The arms AF, BF represent the distances of P and W respectively from the fulcrum.

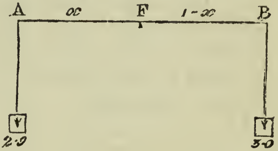


Fig. 44.

Let AF contain x units of length;
∴ FB contains $1-x$ units (for AB $= 1$ metre);

∴ by the principle of the lever above stated,

$$2{\cdot}9 \times x = 3(1-x)\,; \quad \therefore\ x = \frac{3}{5{\cdot}9} \text{ of a metre.}$$

Now $\frac{3}{5{\cdot}9} > \frac{1}{2}$; ∴ the fulcrum must be moved towards A.

For that equal weights should balance each other, it is necessary that the arms should be equal, or the products of units of weight into units of length in arm would not be equal in each case.

∴ the fulcrum must be moved $\left(\frac{3}{5{\cdot}9} - \frac{1}{2}\right)$ metres towards A,

and $\frac{3}{5{\cdot}9} - \frac{1}{2} = \frac{1}{118}$ metres = rather less than a centimetre.

Example 17.—A rigid bar is fixed at one extremity F, and at a distance of $\frac{1}{2}$ metre from F is hung a weight of 2 kilograms. A string attached to the other end passes over a smooth pulley, and supports a weight P. What must be the weight of P that there may be equilibrium, supposing the bar to be 7 metres long?

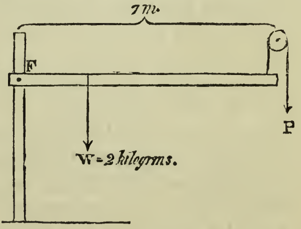


Fig. 45.

Here F is the fulcrum.

The distance of the point of application of P from the fulcrum is 7 metres. That of the application of W is $\frac{1}{2}$ metre;

$$\therefore\ 7P = \tfrac{1}{2} \times 2 \text{ if there is equilibrium;}$$

$$\therefore\ P = \tfrac{1}{7} \text{ kilogram.}$$

Example 18.—In a lever of the third order let the ratio of the weight to the power be as 7 : 9, and the length of the lever be 4 metres. Required the distance of the point of attachment of P from F, that the lever may be in equilibrium.

Let the number of grams in $W = 7a$;

$\therefore$ the number of grams equivalent to $P = 9a$. The distance of the point of application of W from F = length of lever = 4 metres.

Let x = number of metres in distance of point of application of P from F.

$\therefore$ by the principle of levers,

$$x \times 9a = 4 \times 7a;$$

$$\therefore x = \tfrac{28}{9} = 3\tfrac{1}{9}.$$

Therefore P must be applied at a point $3\frac{1}{9}$ metres distant from F.

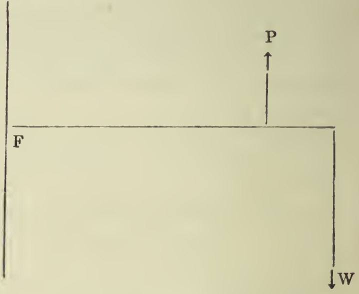


Fig. 46.

(iii.) When the Weight of the Lever is considered.

Thus far we have considered the levers as having no weight. When the lever is supposed to have weight, two points must be kept in mind—that the weight of the lever acts vertically downwards; and that it acts at the centre of gravity of the lever. The centre of gravity, if the lever is uniform, will be at the point of bisection of the lever. In solving questions upon the lever, where the weight of the bar is taken into account, the main point to consider is whether the weight of the lever as a force works with Power or Weight. When this is decided, once more the principle of moments comes into play. If, *e.g.*, the weight of the lever acts with the Power, equilibrium will ensue, when P × its arm + the weight of the lever × the distance between the centre of gravity of the lever and the fulcrum = W × its arm. Or if the weight of the lever act with the Weight or resistance, equilibrium will ensue when W × its arm + the weight of the lever × the distance between the centre of gravity of the lever and the fulcrum = P × its arm.

Example 19.—A uniform lever AB, weighing 10 grams, and measuring 4 metres, has a fulcrum 1 metre from one extremity A. A weight is attached to this extremity. What Power expressed in terms of the Weight applied at the other extremity B would cause equilibrium?

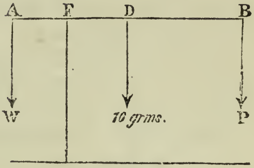


Fig. 47.

Since the bar is uniform, the centre of gravity is at the middle point D.

∴ a weight of 10 grams acts 2 metres from A, and 2 metres from B;

and ∵ AB is 4 metres long,
and AF „ 1 „ „
∴ FD „ 1 „ „

and D is on the same side of F as B (for BF > BD);

∴ the weight of the bar acts with P;

∴ by the principle of levers,

$$W \times 1 = 10 \times 1 + P \times 3;$$

$$\therefore W = 10 + 3P;$$

$$\therefore P = \frac{W - 10}{3} = \frac{W}{3} - 3\tfrac{1}{3} \text{ grams.}$$

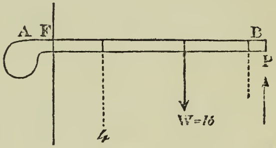


Fig. 48.

Example 20. — A lever of the second order, AB, weighs 4 grams, and has its centre of gravity 8 decimetres from F; 24 decimetres from F a weight of 16 grams is attached.

The lever is kept horizontal by a power equivalent to 16 grams applied at B.

Required the number of decimetres from F to B.

The weight of the lever and that of W act vertically downwards; ∴ they act together.

The Power acts vertically upwards.

Number of dcms. in distance of C. G. from F = 8 cms.
„ „ W „ F = 24 cms.

Let x = number of dcms. in distance of P from F;

∴ by the principle of levers, $4 \times 8 + 16 \times 24 = x \times 16$;

Dividing by 16, ∴ $2 + 24 = x$; ∴ $x = 26$ decimetres.

Exercises.

1. By means of a weightless lever of the second order, whose arms are respectively 2 and 3 metres long, a weight of 60 kilograms is supported by a Power acting at the end of the lever.

Express the power in units of Weight.

2. The arms of a weightless lever of the first kind are 9 and 11 decimetres respectively. What weight at the end of the R shorter arm would balance 4 kilograms 5 hectograms at the end of the longer arm?

3. A heavy lever balances on a support 1 metre from one end, its whole length being 3 metres. If a weight of 20 grams R is hung at the lighter end, the support must be moved to the middle point. What is the weight of the lever?

4. A uniform bar weighing 10 kilograms, and measuring 4 metres, rests upon two supports (A and B), one at each end.

A mass of 20 kilograms is hung 1 metre from B.

Required the pressure on A and B respectively.

5. A lever has a fulcrum at one extremity, and the distance between the points of attachment of P and W is 8 decimetres. If the ratio between the numbers of units of force in P and W is 7 : 6,

(*a*) What is the length of the lever?
(*b*) ,, ,, ,, power-arm?
(*c*) To what order does the lever belong?

CHAPTER IX.

(δ) WHEEL AND AXLE.

(i.) The Simple Wheel and Axle.

THE wheel and axle is a modification of the lever. A large cylinder, the wheel, and a smaller cylinder, the axle, have a common centre, and move round a common axis. To the circumference of the axle or smaller cylinder is attached a cord that carries the Weight; to the circumference of the wheel or larger cylinder is attached a second cord that carries the Power. These two cords are wound round their respective cylinders in opposite directions. If, *e.g.*, the cord supporting the Weight is wound round the axle in the same direction as the hands of a watch move, the cord supporting the Power is wound round the wheel in a direction opposite to that in which the hands of a watch move.

Thus in Fig. 49, showing a vertical section through a wheel and axle, UTC represents the circumference of the wheel, TSA the circumference of the axle. O represents the centre common to both wheel and axle, and through it would pass at right angles to the plane of the paper the axis common to both. Round the axle is wound a cord that carries the Weight W; round the wheel is wound a cord that carries the Power P. The two cords are wound round their respective cylinders in opposite directions. Hence, when P descends and causes the wheel to move round in a direction opposite to that wherein the watch hands move, the cord supporting P unwinds off the wheel. At the same time the axle, turning with the wheel, winds up around it the cord supporting W. This last is therefore raised.

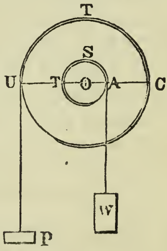


Fig. 49.

The wheel and axle is a modification of the lever. Now a lever is a machine wherein are encountered not only the Power and the Weight, but also a fixed point or fulcrum. The point of application of the Power is at the circumference of the wheel; at U in Fig. 49, at the upper P in Fig. 50. The point of application of the Weight is at the circumference of the axle; at A in Fig. 49, at the upper W in Fig. 50. The fulcrum or fixed point is in the common centre of both wheel and axle; at O in Fig. 49, at F in Fig. 50. Consider Fig. 50; the fulcrum is between the Power and the Weight. The wheel and axle is therefore a lever of the first kind.

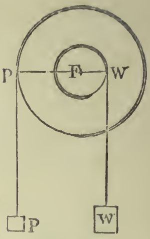


Fig. 50.

Applying to this simple machine the principle of the lever, the algebraical sum of the moment of the Power about the fulcrum, and the moment of the Weight about the fulcrum = zero.

The moment of the Power = P × UO (Fig. 49).

" " Weight = W × AO.

But UO and AO are respectively the radii of the wheel and the axle. Hence there is equilibrium in the wheel and axle when P × radius of wheel = W × radius of axle.

The diameter of a circle = twice the radius. The circumference of a circle $= \pi(\frac{22}{7}) \times$ twice the radius. Diameter $= 2r$; circumference $= 2\pi r$. Hence the diameters or the circumferences of two or more circles bear to one another exactly the same proportion as their radii bear to one another. It must be clearly understood that the diameters and the circumferences of two or more circles are not equal to their radii, but they bear to one another the same ratio as the radii bear to each other.

If a wheel had a radius of 5 decimetres, and an axle a radius of 2 decimetres, the ratio of the two radii $= \frac{5}{2}$.

But as $D = 2r$, the diameter of the wheel = 10 decimetres, and that of the axle = 4 decimetres. The ratio of the two diameters

$= \frac{10}{4} = \frac{5}{2} =$ that of the two radii.

Also, as circumference $= 2\pi r$, the circumference of the wheel $= 10\pi$, and that of the axle $= 4\pi$. The ratio of the two diameters

$= \frac{10\pi}{4\pi} = \frac{10}{4} = \frac{5}{2} =$ that of the two diameters = that of the two radii.

Hence, if the diameters or the circumferences of the wheel and of the axle are given, there is no need to calculate from these the radii. As the radii are *proportional* to the diameters and to the circumferences, the numbers representing the diameters or the circumferences may be taken as representing the two radii. There will be equilibrium when

P × diameter of wheel = W × diameter of axle, or when
P × circumf. ,, = W × circumf. ,, as much as when
P × radius ,, = W × radius ,, .

Example 21.—A wheel of radius $1\frac{1}{2}$ metres has a weight of 4 grams attached to it. What must be the diameter of the axle that a weight of 15 grams attached to it may be exactly balanced by the 4 grams?

Let $P = 4$ grams;
Let $W = 15$ grams.
Let the radius of the wheel $= 1\frac{1}{2}$ metres.

It is required to find the diameter of the axle.

Let the *radius* of the axle be x.

To obtain equilibrium,

$$P \times 1\tfrac{1}{2} = Wx,$$
$$4 \times 1\tfrac{1}{2} = 15x,$$
$$6 = 15x,$$
$$x = \tfrac{6}{15} = \tfrac{2}{5} \text{ m.} = 40 \text{ cm.};$$
$$\therefore \text{diameter} = 80 \text{ cm.}$$

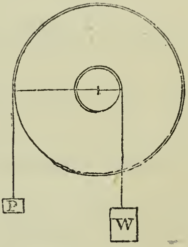


Fig. 51.

(ii.) The Compound Wheel and Axle.

To increase the mechanical advantage of the wheel and axle—*i.e.* to raise a yet larger Weight by a yet smaller Power—two things may be done. The wheel may be increased in size, and the arm, and therefore the moment of the Power increased; or the axle may be diminished in size, and the arm, and therefore the moment of the Weight diminished. But both of these methods after a certain limit become impracticable. If the wheel is greatly increased in size, it becomes cumbersome and unmanageable; if the axle is exceedingly diminished in size, it becomes of insufficient strength to support the Weight.

Hence the compound wheel and axle has been devised. In this arrangement the Power is attached, as in the simple wheel and axle, to the wheel. The Weight is supported by a single moveable pulley. The cord, passing underneath this pulley, passes up on the side remote from the Power to the larger of the *two* axles OC; the other part of the cord passes up on the side adjacent to the Power to

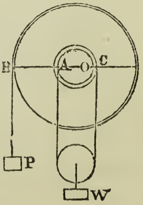


Fig. 52.

the smaller of the two axles OA. Each of these cords supports one-half W or $\frac{W}{2}$. If we consider P as descending and the wheel moving round in a direction opposed to that wherein the hands of a watch move, the cord supporting the Weight will be wound up on the larger axle, and at the same time wound off the smaller axle. On the whole, therefore, the Weight will rise.

If we take moments about the fixed centre O, we have

$$P \times BO + \frac{W}{2} \times AO = \frac{W}{2} \times CO.$$

$$P \times BO = \frac{W}{2}(CO - AO).$$

P × the radius of the wheel $= \frac{W}{2} \times$ the *difference* between the radius of the large axle and the radius of the small axle.

Hence, by making the *difference* between the two axles very small, a great mechanical advantage can be obtained without either using a cumbersome wheel, or an axle so small as to be unable to support a heavy weight.

Example 22.—In the compound wheel and axle, when the circumference of the wheel is $50\frac{2}{7}$ decimetres, and the radii of the two axles respectively 10 centimetres and $9\frac{4}{5}$ centimetres, what is the Power when the Weight = 8 hectograms?

Let the circumference of the wheel $= 50\frac{2}{7}$ decimetres.

$$50\tfrac{2}{7} = 2\pi r; \therefore r = \frac{\frac{352}{7}}{\frac{44}{7}} = \frac{352}{44} = 8;$$

∴ the radius = 8 dcm. = 80 cm.

Let the radius of the larger axle = 10 cm.

Let the radius of the smaller axle $= 9\frac{4}{5}$ cm.

Let W = 8 hectograms = 800 grams; it is required to find P.

Now P × radius of wheel $= \frac{W}{2} \times$ the difference between the radii of the axles.

$$P \times 80 = 400 \text{ grams} \times \tfrac{1}{5},$$
$$80P = 80 \text{ grams},$$
$$P = 1 \text{ gram}.$$

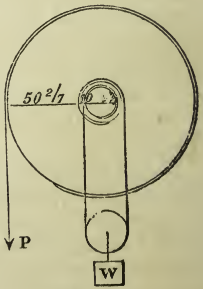


Fig. 53.

EXERCISES.

1. A wheel of radius 7 decimetres 2 centimetres has an axle of 6 centimetres radius. Find W when P = 1 gram.

2. What is the diameter of the axle when the radius of the wheel being 1 decimetre 6 centimetres, P = 2 and W = 20½ grams?

3. Find the circumference of the axle and of the wheel when the radius of the axle being 5 centimetres, P = 14 grams, W = 70 grams.

4. The circumference of an axle is $18\frac{6}{7}$ centimetres, the radius of the wheel = 1 decimetre. If P = 3, find W.

5. If the circumference of an axle = $12\frac{4}{7}$ centimetres, and the diameter of the wheel = 5 decimetres 8 centimetres, find ratio of P to W.

6. W = 10½P. Find how often the circumference of the axle is contained in that of the wheel.

CHAPTER X.

(*c*) TOOTHED WHEELS.

This form of simple machine is of value for connecting different parts of machinery together. A toothed wheel is a circular plate, usually of metal, and with its circumference cut into teeth. The teeth must be all equal one to the other, not only on the one wheel, but on both; that is, the teeth on one wheel must be all of the same size, and also all of the same size as those on the other wheel. The distances between them must also be the same on both wheels. Hence the number of teeth on each wheel will be proportional to the circumference of each wheel, and therefore to the radius or to the diameter of each. If the teeth are small, it is allowable to regard the pressure of the one wheel upon the other as a constant in magnitude and direction. Let us call this constant pressure of the one upon the other C. The axles of the two wheels are always equal. Denote their radii by r, and those of the two wheels by R and R′, taking R as that of the wheel to whose axle the Power is attached, R′ as that of the wheel to whose axle the Weight is attached.

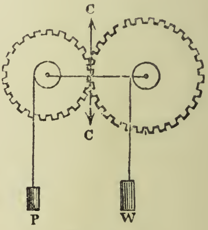


Fig. 54.

Now, since P and the constant pressure exerted by the two wheels, one on the other, are in equilibrium, the algebraical sum of their moments about the centre of the left-hand wheel = zero, or $Pr = CR$. In like manner, as W and the constant pressure exerted by the two wheels, one on the other, are in equilibrium, the algebraical sum of their moments about the centre of the right-hand wheel = zero, or $Wr = CR'$.

But if $Pr = CR$, and $Wr = CR'$, by cross multiplication,

$$PrCR' = WrCR.$$

Dividing by rC,

$$PR' = WR.$$

Power × radius of the Weight's wheel = Weight × radius of the Power's wheel.

Or Power × circumference of the Weight's wheel = Weight × circumference of the Power's wheel.

Or Power × number of teeth in the Weight's wheel = Weight × number of teeth in the Power's wheel.

Example 23.—If the radii of two toothed wheels with equal axles are respectively 7 and 4 decimetres, and to the axle of the smaller is applied a pressure = 40 grams; find what weight can be supported on the axle of the larger.

Let P = pressure on axle of small wheel. Let P' = weight on axle of large wheel. Let R and R' = radii of small and large wheels respectively.

$P = 40$ grams. It is required to find P'.

$$\begin{aligned} PR' &= P'R, \\ 40 \times 7 &= P'4; \\ \therefore\ 10 \times 7 &= P', \\ 70 &= P'. \end{aligned}$$

EXERCISES.

1. How many teeth are cut in the circumference of a wheel, when, with a hectogram attached to its axle, it is in equilibrium with another toothed wheel that has 40 teeth cut in its circumference, and has a weight of 16 grams attached to its axle?

2. Two toothed wheels, having respectively 65 and 35 teeth, balance when 70 grams are attached to the axle of the larger, and an unknown weight to the axle of the smaller. What is the unknown weight?

3. A wheel of circumference 3 decimetres has 30 teeth cut in its circumference, and a weight of 4 grams attached to its axle. It works upon another wheel of radius 4 decimetres, $7\frac{8}{11}$ centimetres, in whose circumference teeth are cut of the same size as in the first. What weight must be attached to the axle of this latter wheel that equilibrium may ensue, the teeth being of the same size as the intervals between them?

4. A wheel has teeth one centimetre broad, with intervals of one centimetre between them. Its radius is 14 decimetres, and the weight on its axle 10 grams. It works on another wheel whose teeth are of the same size, and intervals of the same breadth as in the first. One gram is attached to the axle of the second wheel. Find its diameter if equilibrium obtains.

CHAPTER XI.

(*d*) THE PULLEY.

The third simple machine, or the pulley, is a modification of the lever. A pulley is a wheel with a cord or band passing round its circumference, this latter being often grooved to admit of the better 'biting' of the cord. Theoretically, the cord or band is perfectly flexible, perfectly smooth, and perfectly weightless; the pulleys, also, are at first to be considered as weightless. Under these theoretical conditions, the effect of friction between the cord and the pulley may be neglected, and the weights of the cords and the pulleys may be left out of consideration. A necessary structure in connection with the working of all pulleys is the beam. The beam is the fixed rod or other structure to which the pulleys are connected directly or indirectly. Thus, in pulling up a window-blind, the pulley at the top of the window frame, over which passes the window cord, is fixed into the solid wall or solid framework up and down which the window can be moved. It will be found that the beam is of great use in supporting more or less of the Weight or resistance. When a small Power in this simple machine balances or overcomes a greater Weight, some of that Weight is supported by the beam. In fact, the beam supports just so much of the Weight as would be left after subtracting the Power from the Weight. Further, when the pulleys are regarded as having weight, their weight is in most cases also partially supported by the beam.

(i.) **Pulleys and cord weightless: no friction.**

(*a*) Fig. 55 represents a single pulley, with iron or wooden bars attached to a pivot passing through its centre. These bars pass upwards to be attached to the beam above. A cord passes over the pulley. To one end of the cord is attached a Weight, say of 10 grams. The Power applied at the other end of that cord must be equal to the Weight if there is equilibrium, or in this case equal 10 grams. Therefore $P = W$. There is in this case no gain of power; the mechanical advantage $= \frac{W}{P} = \frac{10}{10} = 1$. But there is a gain in this way. The Power acting downwards can move the Weight *upwards*, if that Power be in the least greater than the Weight. It would be very awkward if, on every occasion when we desired to draw up a window-blind, we had to climb to the top of the window, and pull upwards. It is on the whole more convenient to stand on the customary floor, and exerting force *downwards*, to pull the blind *upwards*; and this is the value of the fixed pulley. A fixed pulley, then, is one whose centre, or axle on which the pulley moves, only is fixed or attached to the beam, and its use is to change the direction of a force, so that a Power acting downwards or upwards may overcome a Weight or resistance acting upwards or downwards.

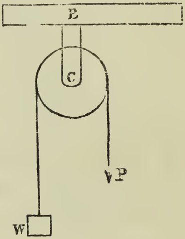


Fig 55.

Example 24.—With a single fixed pulley, if the friction between the rope and the pulley may be regarded as equivalent to $\frac{1}{10}$ of the Weight, what is the ratio of the Power to the Weight, and also to the friction?

Let the friction F between the rope and the pulley $= \frac{1}{10}W$

It is required to find the ratio of P to W and of P to F.

The friction will act against the Weight, and will sustain $\frac{1}{10}$ of the Weight.

With a fixed pulley, $P = W$.

But here P available $= P + \frac{W}{10}$;

$$\therefore \; P + \frac{W}{10} = W;$$

$$\therefore \; 10P + W = 10W;$$

$$\therefore \; 10P = 9W,$$

$$\text{and } P = \frac{9}{10}W;$$

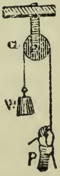


Fig. 56.

$$\therefore \ \frac{P}{W}=\frac{9}{10}, \text{ or } P : W :: 9 : 10,$$

$$\text{and } \frac{P}{F}=\frac{\frac{9}{10}}{\frac{1}{10}}, \text{ or } \frac{9}{1}, \text{ or } P : F :: 9 : 1.$$

(β) Fig. 57 represents a single pulley with no direct connection with the beam. A cord passes under the pulley. To the axle of the pulley is attached a weight, say of 10 grams. The Power is applied at one end of the cord, and the other end of the cord is affixed to the beam. The Weight is therefore supported by two parts of the cord, viz. AB going up to the beam, and CP going up to the Power. Each of these parts of the cord supports one-half of the Weight, *i.e.* 5 grams in this particular case. Hence the beam supports 5 grams, and the Power has to support the other 5 grams if there is equilibrium; $\therefore$ $P=\frac{W}{2}$; or in this case there is gain of power: the mechanical advantage $=\frac{W}{P}=\frac{10}{5}=2$. A Power = one-half the Weight supports that Weight. And this is the value of a moveable pulley. A moveable pulley, then, is one whose centre or axle is not fixed or attached to the beam, and its use is to enable a Power to support a Weight equal to double that Power, or to enable one to support a Weight by a Power equal to one-half the Weight.

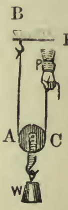


Fig. 57.

Example 25.—With a single moveable pulley, whose weight may be neglected, if the Power is equal to $2\frac{1}{2}$ hectograms, find the Weight supported in fractions of a kilogram.

Let $P = 2\frac{1}{2}$ hectograms, or $\frac{250}{1000}$ or $\frac{1}{4}$ kilogram.
It is required to find W in fractions of a kilogram.
Half W is supported by the beam and half by P.
But $P = \frac{1}{4}$ kilogram;

$$\therefore \ \frac{W}{2}=\frac{1}{4} \text{ kilogram};$$

$$\therefore \ W = 2 \times \frac{1}{4} \text{ kilogram} = \frac{1}{2} \text{ kilogram}.$$

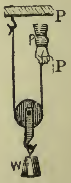


Fig. 58.

(γ) Fig. 59 represents four pulleys. A cord passes downwards from the beam under the lowest pulley, and then up to the axle of the lowest pulley but one. This cord is represented by the line BACO. A second cord passes downwards from the beam under the lowest pulley but one, and then up to the axle of the lowest pulley but two. This cord is represented by the line B′DEO′. A third cord passes downwards from the beam under the lowest pulley but two, and then upwards over the topmost pulley, and has the power affixed to its free end. This cord is represented by the line B″FGHKP. The three lowest pulleys are moveable, for their centres are not attached to the beam. The topmost pulley is a fixed one, for the pivot passing through its centre is attached to the beam.

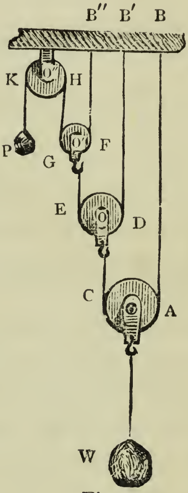


Fig. 59.

Consider the lowest pulley. It carries a weight W. Two pieces of cord support this weight, viz. AB and CO. Each of them carries one-half of W; ∴ up the cord AB is transmitted pressure $= \frac{W}{2}$ *to the beam*, and up the cord CO is transmitted pressure $= \frac{W}{2}$ to the second pulley (counting from below). Hence the pulley with centre O has dragging upon it a force acting downwards $= \frac{W}{2}$. But two pieces of cord have to sustain this downward drag $= \frac{W}{2}$, viz. DB′ and EO′. Each of these carries one-half of $\frac{W}{2}$ or $\frac{W}{4}$; ∴ up the cord DB′ is transmitted one-half $\frac{W}{2}$ or $\frac{W}{4}$ *to the beam*, and up the cord EO′ is transmitted one-half $\frac{W}{2}$ or $\frac{W}{4}$ to the third pulley (counting from below). Hence the pulley with centre O″ has dragging upon it a force acting downwards $= \frac{W}{4}$. But two pieces of cord have to sustain this downward drag $=\frac{W}{4}$, viz. FB″ and GH. Each of these carries one-half of $\frac{W}{4}$ or $\frac{W}{8}$; ∴ up the cord FB″ is transmitted one-half $\frac{W}{4}$ or $\frac{W}{8}$ *to the beam*, and up the cord GH is transmitted one-half $\frac{W}{4}$ or $\frac{W}{8}$ to the fourth pulley or the uppermost pulley. But this topmost pulley is a fixed pulley, and we have seen that the fixed pulley gives no mechanical advantage, but only changes the direction of the force; ∴ the pulley with the centre O″ only changes the direction of the force, and $P=\frac{W}{8}$. And this is the first system of pulleys wherein all the pulleys but the top one are moveable pulleys: wherein each of the moveable ones has a separate cord fixed by one end to the beam above, passing under the pulley and attached by its other end to the axle of the next moveable pulley above. In the case of the top *moveable* pulley, the cord still fixed by one end to the beam above, and still passing under the pulley afterwards, passes over the fixed pulley and has the Power affixed to its free end. Each moveable pulley halves the Weight, and the fixed pulley changes the direction. With one moveable pulley, $P=\frac{W}{2}$; with two moveable pulleys, $P=\frac{W}{4}$ or $\frac{W}{2^2}$; with three moveable pulleys, $P=\frac{W}{8}$ or $\frac{W}{2^3}$; with n moveable pulleys, $P=\frac{W}{2^n}$. It should be noticed that just so much of W as P does *not* exactly equal is borne by the long-suffering beam. Thus in the case we are studying, $P=\frac{W}{8}$. This leaves $\frac{7}{8}$ of W to be supported by the beam.

But cord AB transmits to beam $\frac{W}{2}$ or $\frac{4}{8}$ of W;
" DB′ " " $\frac{W}{4}$ or $\frac{2}{8}$ of W;
" FB″ " " $\frac{W}{8}$ or $\frac{1}{8}$ of W;

$\therefore$ the beam supports $\frac{4}{8}+\frac{2}{8}+\frac{1}{8}=\frac{7}{8}$ of W.

Example 26.—If in an arrangement of pulleys according to the first system, the Weight is 1 kilogram, find the Power required to support that Weight when you have only five pulleys (weightless) at your disposal.

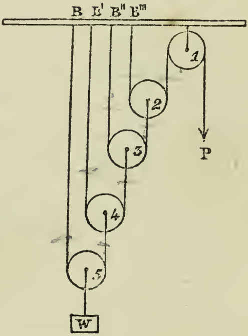


Fig. 60.

Let 1 be a fixed, 2, 3, 4, 5 moveable pulleys. Let W = 1 kilogram.

It is required to find P.

Pulley 5 sustains W, and transmits $\frac{1}{2}$ kilogram to B, and $\frac{1}{2}$ kilogram to pulley 4.

Pulley 4 sustains $\frac{W}{2}=\frac{1}{2}$ kilogram, and transmits $\frac{1}{4}$ kilogram to B′, and $\frac{1}{4}$ kilogram to pulley 3.

Pulley 3 sustains $\frac{W}{4}=\frac{1}{4}$ kilogram, and transmits $\frac{1}{8}$ kilogram to B″, and $\frac{1}{8}$ kilogram to pulley 2.

Pulley 2 sustains $\frac{W}{8}=\frac{1}{8}$ kilogram, and transmits $\frac{1}{16}$ kilogram to B‴, and $\frac{1}{16}$ kilogram over pulley 1. This last, being fixed, only alters the direction of the force.

$$\therefore\ P=\frac{W}{16}, \text{ or } \frac{1}{16} \text{ kilogram} = 62\tfrac{1}{2} \text{ grams.}$$

Example 27.—Three moveable pulleys (weightless) are arranged according to the first system. With a Power of $\frac{1}{2}$ gram, what Weight could be sustained?

Let 1 be a fixed, 2, 3, and 4 moveable pulleys.

Let $P=\frac{1}{2}$ gram. It is required to find W.

P is transmitted by 1, unaltered in amount, altered in direction.

As there is a tension of P in each of the two parts of the cord supporting pulley 2; $\therefore$ pulley 2 transmits twice $P = 2P$ to pulley 3.

In like manner, pulley 3 transmits twice $2P = 4P$ to pulley 4.

Pulley 4 transmits twice $4P = 8P$ to W.

But $P = \frac{1}{2}$;

$$\therefore\ W = 8 \times \tfrac{1}{2} = 4 \text{ grams.}$$

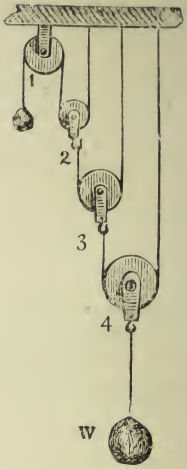


Fig. 61.

Example 28.—With four weightless pulleys only at your disposal, to be arranged according to the first system, find the Power and the pressure upon the beam when the Weight = 2 hectograms.

Let 1 be a fixed, 2, 3, 4 moveable pulleys.

Let W = 2 hectograms. It is required to find P and the pressure on the beam.

The weight W is supported by B and pulley 3;

$\therefore$ 1 hectogram is the pressure on B; 1 hectogram is transmitted to pulley 3.

Of this 1 hectogram transmitted to pulley 3, $\frac{1}{2}$ hectogram is pressure on B′, and $\frac{1}{2}$ hectogram is transmitted to pulley 2.

Of this $\frac{1}{2}$ hectogram transmitted to pulley 2, $\frac{1}{4}$ hectogram is pressure on B″, and $\frac{1}{4}$ hectogram is transmitted over pulley 1 and = P;

$$\therefore\ P = \tfrac{1}{4} \text{ hectogram} = 25 \text{ grams.}$$

Pressure on beam

$= 1$ hectogram $+ \frac{1}{2}$ hectogram $+ \frac{1}{4}$ hectogram

$= 100$ grams $+ 50$ grams $+ 25$ grams

$= 175$ grams.

It will be seen that P + pressure on beam.

$25 + 175 = W$ or 200.

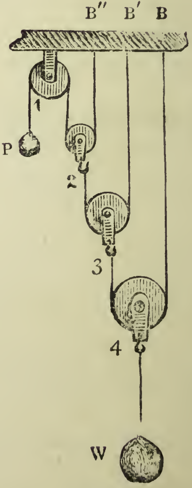


Fig. 62.

(δ) Fig. 63 represents six pulleys. One cord only is used. This cord is attached by one end to the Power. It passes upwards over pulley 1, downwards under pulley 2, upwards over pulley 3, downwards again under pulley 4, upwards over pulley 5, downwards under pulley 6, upwards to be attached to the hook affixed to pulley 5. The Weight is affixed to the bar joining the pivots passing through the centres of the lower set of pulleys (2, 4, 6). Now the Weight is supported by six pieces of cord (count them between pulleys 5 and 6), and each of these carries $\frac{1}{6}$ of W ;

∴ the tension in the cord running from pulley 1 to $P = \frac{W}{6}$, and $P = \frac{W}{6}$.

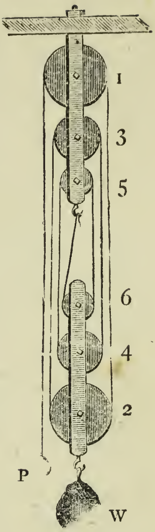


Fig. 63.

And this is the second system of pulleys wherein all the upper pulleys are fixed, all the lower are moveable. One cord only is used, and the pieces of that cord supporting the Weight are twice as many in number as the number of moveable pulleys in the lower set. With one such moveable pulley, $P = \frac{W}{2}$; with two such moveable pulleys, $P = \frac{W}{4}$; with three such moveable pulleys, $P = \frac{W}{6}$; with n such moveable pulleys, $P = \frac{W}{2n}$.

Fig. 64 represents another arrangement of similar nature to the last, belonging to the second system of pulleys. Here the three pulleys of each 'block' or set are all on the same level. The same reasoning as in the last case obtains.

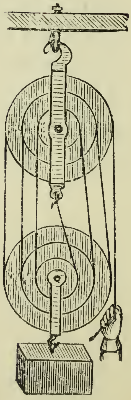


Fig. 64.

Example 29.—In the second system of pulleys, with five pulleys (weightless) in the upper 'block' or set, what Power will support a Weight of 3 decagrams?

With five pulleys in the upper block, five are required in the lower.

Let 1, 3, 5, 7, 9 be the fixed upper, and 10, 8, 6, 4, 2 the moveable lower pulleys.

Let W = 3 decagrams. It is required to find P.

There are 10 parts of the string, in which the tension is equal; ∴ each part has tension $= \frac{1}{10}$ of 3 decagrams $= \frac{3}{10}$ decagram = 3 grams;

$$\therefore\ P = 3 \text{ grams.}$$

Or, $P = \dfrac{W}{2n}$; n or number of moveable pulleys $= 5$;

$$\therefore\ P = \frac{W}{2n} = \frac{W}{10} = \frac{3}{10} = 3 \text{ grams.}$$

So far

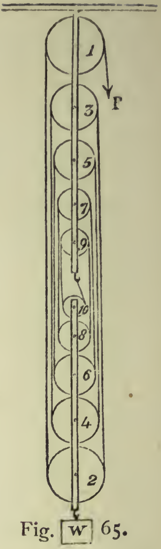


Fig. 65.

Example 30.—With 8 pulleys (weightless) at your disposal, to be arranged according to the second system, if P = 3 grams, find W.

The 8 pulleys must be 4 fixed and 4 moveable.

Let 1, 3, 5, 7 be the fixed, and 8, 6, 4, 2 the moveable pulleys.

Let P = 3 grams.

It is required to find W.

There are eight parts of the string, and the tension in each is equal; but in string from P to pulley 1, tension = 3 grams; ∴ tension in all = 3 × 8 = 24 grams;

$$\therefore\ W = 24 \text{ grams.}$$

$$\text{Or, } P = \frac{W}{2n} \text{ and } n = 4;$$

$$\therefore\ 3 = \frac{W}{8},$$

$$\text{and } 24 = W.$$

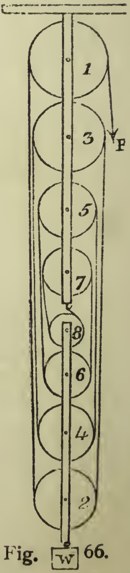


Fig. 66.

Example 31.—How many pulleys would be required to make an arrangement on the second system, whereby a Power = 2 grams could sustain a Weight = 40 grams?

Let $P = 2$ grams, and let $W = 40$ grams.
It is required to find n.

$$P = \frac{W}{2n},$$

$$2 = \frac{40}{2n},\ 4n = 40;$$

$$\therefore\ n = \frac{40}{4} = 10;$$

$\therefore$ 10 moveable pulleys would be required.
But the number of moveable pulleys = number of fixed pulleys;
$\therefore$ 20 pulleys in all would be required.

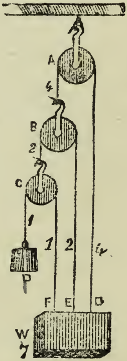


Fig. 67.

(ϵ) Fig. 67 represents yet another arrangement. The Weight is supported by three cords at F, E, D. To the point D is attached a cord passing upwards, and over the highest pulley A, to be attached to the axle of the second pulley B. From E in the Weight, a second cord passes upwards over pulley B, to be attached to the axle of the third pulley C. From F a third cord passes over C. To the free end of this cord is attached the Power P.

In the cord passing over pulley C is a tension $= P$; $\therefore$ the force acting upwards at $F = P$. As there is a tension $= P$ in *both* parts of the cord PCF, *i.e.* in the part that runs from P to C, and in the part that runs from C to F, the tension in the cord that runs from C to $B = 2P$; $\therefore$ the force acting upwards at $E = 2P$. As there is a tension $= 2P$ in *both* parts of the cord CBE, *i.e.* in the part that runs from C to B, and in the part that runs from B to E, the tension in the cord that runs from B to $A = 4P$; $\therefore$ the force acting upwards at $D = 4P$.

The sum of the forces acting upwards at F, E, D respectively $= P + 2P + 4P$, or $P(1 + 2 + 4)$, or $7P$.

And this is the third system of pulleys wherein there is a separate cord passing over each pulley, and joined by one end to the bar that carries the weight, by the other to the pulley below, or, in the case of the cord that passes over the lowest pulley, to the Power. With one pulley thus arranged, we should have only a single fixed pulley, and $W = P$; with two pulleys, $W = P + 2P$ or $P(1 + 2)$; with three pulleys, $W = P + 2P + 4P =$

$P(1+2+4)$; with four pulleys, $W = P + 2P + 4P + 8P = P(1 + 2 + 4 + 8)$; with n pulleys, $W = P \times n$ terms of the series $1 + 2 + 4 + 8 +$ etc.

As the upward tensions at F, E, D are not equal, the centre of gravity of W must not be under E, but in such a position that the algebraical sum of the moments of the three forces acting upwards at F, E, D respectively, may, when calculated about the centre of gravity of the Weight, equal zero.

Example 32. — Third system of pulleys. With a Power = 1 gram, and four (weightless) pulleys in use, find the Weight.

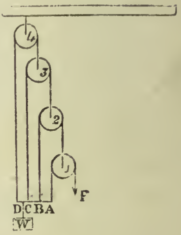


Fig. 68.

Let 1, 2, 3, 4 be 4 pulleys. Let string from P pass over 1, and be attached at A. Let string attached to axle of 1 pass over 2, and be attached at B. Let string attached to axle of 2 pass over pulley 3, and be attached at C. Let string attached to axle of 3 pass over pulley 4, and be attached at D. Let axle of 4 be attached to beam.

Let P = 1 gram. It is required to find W.

1 gram acting at P transmits to A a tension = 1 gram.

1 gram tension in each part of string passing over 1 causes transmission to B of a tension = 2 grams.

2 grams tension in each part of string passing over 2 causes transmission to C of a tension = 4 grams.

4 grams tension in each part of string passing over 3 causes transmission to D of a tension = 8 grams.

$\therefore$ tensions supporting $W = P + 2P + 4P + 8P$, or $P(1 + 2 + 4 + 8) = 15P = 15$ grams; $\therefore$ W supported = 15 grams.

Example 33.—Five pulleys without weight are arranged on the third system. Find P if $W = 155$.

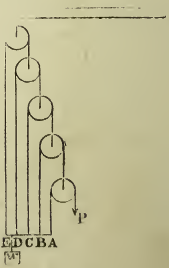


Fig. 69.

Let A, B, C, D, E be the points of a rod to which is suspended W, a weight of 155 grams. It is required to find P.

Applying the principle involved in the preceding answer,

$$P(1 + 2 + 4 + 8 + 16) = 155,$$
$$31P = 155,$$
$$P = \frac{155}{31} = 5 \text{ grams.}$$

Example 34.—How many weightless pulleys are needed, arranged on the third system, that a Power of 2 may balance a Weight of 254?

Let $P = 2$, $W = 254$.

On the third system, $W = P(1 + 2 + 4 + 8 + \text{etc.})$;

$$\therefore\ 254 = 2(1 + 2 + 4 + 8 + \text{etc.});$$

$$\therefore\ \frac{254}{2} \text{ or } 127 = (1 + 2 + 4 + 8 + \text{etc.}).$$

But $1 + 2 + 4 + 8 + \text{etc.}$ are a series of quantities in geometrical progression. Reference to any standard work on algebra will show the student that

if a represent the first term of such a series,
r ,, common ratio or multiplying number,
n ,, number of terms,
S ,, sum of n terms.

$S = \dfrac{a(r^n - 1)}{r - 1}$. Therefore in this especial case, where $a = 1$, $r = 2$, n is unknown, $S = 127$:

$$127 = \frac{1(2^n - 1)}{2 - 1} = \frac{2^n - 1}{1};$$

$$\therefore \text{ cross multiplying, } 127 = 2^n - 1;$$

$$\therefore\ 128 = 2^n;$$

$$\therefore\ n = 7.$$

Hence seven pulleys are required.

Thus far we have studied the fixed pulley, whose value is to change the direction of force; the moveable, which enables us to support a Weight by a Power = one-half the Weight; the first system of pulleys, where each pulley has a separate cord passing from the beam downwards, under the pulley, and up to the axle of the next pulley immediately above; the second system, where only one cord is used that passes over the successive pulleys of the upper block, and under the successive pulleys of the lower block; the third system, where each pulley has a separate cord which passes from the next pulley immediately below, or in the case of the lowest pulley from the Power, over the pulley under consideration, and down to the bar that carries the Weight. It remains to consider these various arrangements when the weight of the pulleys is taken into account.

(ii.) Pulleys and Cords having weight: no friction.

(a) *Fixed pulley.*—The weight of the pulley and of the cord in this case will in no way affect the Power, as these weights will be borne by the beam.

(β) *Moveable pulley.*—The weight of the pulley and of the cord will affect the Power. Instead of considering the Weight only as to be supported, we must consider the Weight + the weight of the pulley + the weight of the cord. The sum of all these three has to be sustained. One-half that sum will be supported by the beam, the other half by the Power, which therefore $= (W + \textit{weight of pulley} + \textit{weight of cord}) \div 2$.

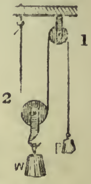


Fig. 70.

Example 35.—With one moveable and one fixed pulley each weighing 6 grams, find W when P = 20 grams.

Let P = 20 grams.

Pulley 1 = fixed pulley weighing 6 grams,
,, 2 = moveable ,, 6 grams.

It is required to find W.

Weight of 1 is supported entirely by the beam, and may therefore be disregarded. The cord passing under 2 has to support W + 6 grams. Each part of the cord supports therefore $\frac{W+6}{2}$;

$$\therefore 20 = \frac{W+6}{2};$$
$$\therefore 40 = W + 6;$$
$$\therefore 40 - 6 = W,$$
$$34 = W.$$

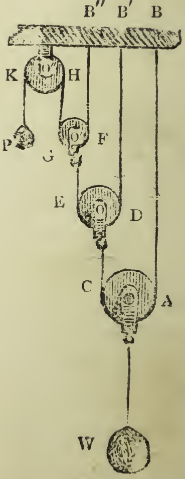


Fig. 71.

(γ) *First System.*—The weights of all the moveable pulleys must be taken into account; that of the fixed pulley need not be considered, as it is supported entirely by the beam. Suppose the weights of the pulleys AC, DE, FG are represented respectively by the letters w, w', w''. The cord BACO has to support not only W (the Weight attached to AC), but also w (the weight of AC); ∴ each part of the cord BACO carries $\frac{W+w}{2}$. The cord B′DEO′ has to support not only $\frac{W+w}{2}$, but also w' (the weight of the pulley DE); ∴ each part of the cord

B′DEO′ carries $\frac{W+w}{4}+\frac{w'}{2}$. The cord B″FGH has to support not only $\frac{W+w}{4}+\frac{w'}{2}$, but also w'' (the weight of the pulley FG); ∴ each part of the cord B″FGH carries $\frac{W+w}{8}+\frac{w'}{4}+\frac{w''}{2}$, and so on with any number of moveable pulleys.

Example 36.—With five pulleys arranged upon the first system, their weights from below upwards being respectively 3, $2\frac{1}{2}$, 4, $5\frac{1}{2}$, 3 grams, find what Power would support a Weight = 96 grams.

Refer to Fig. 71, where there are only four moveable pulleys, but the principle of reasoning is the same as in the case where there are five.

Let there be 5 pulleys weighing respectively 3, $2\frac{1}{2}$, 4, $5\frac{1}{2}$, 3 grams: let W = 96 grams. It is required to find P.

$\frac{96+3}{2}$ are supported by the beam,

and $\frac{96+3}{2}$, or $49\frac{1}{2}$ by second pulley.

Again, $49\frac{1}{2}+2\frac{1}{2}$ are supported by the beam and the third pulley

∴ $\frac{52}{2}$ are supported by the beam and $\frac{52}{2}$, or 26 by third pulley.

Again, 26 + 4 are respectively supported by beam and fourth pulley;

∴ $\frac{30}{2}$ are supported by beam and $\frac{30}{2}$, or 15 by fourth pulley.

Again, $15+5\frac{1}{2}$ are supported by beam, and by the cord passing over fixed pulley to P;

∴ $\frac{20\frac{1}{2}}{2}$ are supported by beam and $\frac{20\frac{1}{2}}{2}$,

or $10\frac{1}{4}$ are supported by P.

The weight of fixed pulley is supported by beam;

∴ $10\frac{1}{4}$ are supported by P;

∴ P = $10\frac{1}{4}$ grams.

(δ) *Second System.*—The weights of the moveable pulleys or all those in the lower block must be considered. The weights of the fixed pulleys, or all those in the upper block, only affect the beam and do not affect the Power at all.

Suppose the weights of the pulleys marked 2, 4, 6, Fig. 63, to be respectively w, w', w''. The whole weight to be sustained is therefore the Weight attached to the lower block (W) + $w+w'+w''$.

Six cords support this compound weight; each carries $\frac{1}{6}$ thereof;

$$\therefore P = \frac{W + w + w' + w''}{6}.$$

Example 37.—With ten pulleys to be arranged upon the second system, and Weight = 35 grams, find the weight of the lower block when the Power = 4 grams.

Let $P = 4$ grams; let $W = 35$ grams; let w = weight of lower block. It is required to find w.

As there are 5 moveable pulleys, $P = \frac{1}{10}$ of whole weight supported.

$$\tfrac{1}{10} \text{ of whole weight} = 4;$$
$$\therefore \text{ whole weight} = 40;$$
$$\therefore W + w = 40.$$
$$\text{But } W = 35; \therefore w = 5.$$

(ϵ) *Third System.*—Suppose the weights of the pulleys C, B, A to be respectively w, w', w''. At the point F is an upward force = P. In the cord going from C over B to E, there will be not only a tension = 2P, but also an additional tension = weight of C; ∴ the upward force acting at $E = 2P + w$. In the cord going from B over A to D, there will be not only a tension $= 2(2P + w)$, but also an additional tension $= w'$; ∴ the upward force acting at $D = 2(2P + w) + w'$. The weight of pulley A will not affect the result, as this pulley is a fixed one.

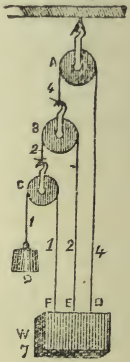


Fig. 72.

Example 38. — Four pulleys are used weighing from below upwards 2, 3, 4, 5 grams; if the Power = 1 gram, find the Weight (third system).

Let m, n, o, p be four pulleys weighing respectively 2, 3, 4, 5 grams.

Let $P = 1$ gram. It is required to find W.

Tension in cord over $m = P = 1$ gram.

" " " $n = 2 \times 1 +$ weight of $m = 2 + 2 = 4$ grams.

Tension in cord over $o = 2 \times 4 +$ weight of $n = 8 + 3 = 11$ grams.

Tension in cord over $p = 2 \times 11 +$ weight of $o = 22 + 4 = 26$ grams.

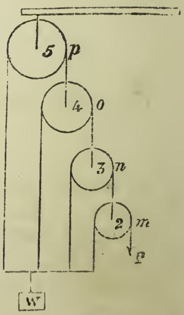


Fig. 73.

Weight of *p* does not affect result, as it is fixed to the beam. Total tension upwards $= 1 + 4 + 11 + 26 = 42$ grams $=$ W.

(iii.) **The Pulley regarded as a Lever.**

It has been said that the pulley is a modification of the lever. Now a lever must have in addition to the Power (P) and the Weight (W), a fixed point or fulcrum (F). In our two kinds of pulley, fixed and moveable, we must find a Power, a Weight, and a fulcrum, if we are to show that they are modifications of the lever.

In the fixed pulley represented to the left of Fig. 74, the point of application of the Power is at one side of the pulley, that of the Weight is at the other, and the centre of the pulley is the fulcrum. The fulcrum is therefore between the points of application of the Power and the Weight, and the fixed pulley is a lever of the first kind.

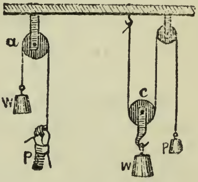


Fig. 74.

In the moveable pulley represented, with a fixed one, to the right of the same figure, the point of application of the Power is at one side of the pulley *c*, that of the Weight is at the centre of the moveable pulley *c*, and the fulcrum or fixed point is in the beam actually, but may be regarded, as far as the pulley is concerned, as at the opposite side to P. The point of application of the Weight, therefore, is between the fulcrum and the point of application of the Power, and the moveable pulley is a lever of the second kind.

(iv.) **Special Cases.**

(*a*) *With two fixed pulleys, whose diameters and the horizontal distance between whose circumferences are all equal.* The pulleys 2 and 3 (Fig. 75) represent these. Number 1 is a moveable pulley, of diameter = twice that of either of the fixed ones. Three pieces of cord support 1 and the attached Weight. Each of them carries $\frac{1}{3}$ of (W + weight of pulley 1); $\therefore$ P $= \frac{1}{3}$ of (W + weight of pulley 1).

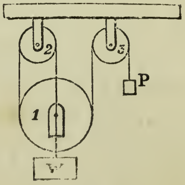


Fig. 75.

Example 39. — In an arrangement of pulleys such as is represented in Fig. 75, what must be the Weight if the weight of the moveable pulley is 3 grams, and the Power = 8 grams?

Let pulley 1 weigh 3 grams. P = 8 grams.

It is required to find W.

W + 3 are supported by 3 cords;

$$\therefore \text{ tension in each cord} = \frac{W+3}{3};$$

$$\therefore P = \frac{W+3}{3}, \text{ and } 3P = W + 3.$$

$$\text{But } P = 8 \text{ grams}; \therefore 24 = W + 3,$$

$$24 - 3 = W,$$

$$21 = W.$$

(β) *With a man as Weight and his own muscular effort as Power.*—In Fig. 76, pulley 1 is a moveable pulley, pulley 2 a fixed one. The man is suspended from pulley 1. If he grasp the free end of the single cord that goes under pulley 1 and over pulley 2, and exert sufficient force to balance himself, part of his own weight is now the Power. Let this be represented by P. Hence pulley 1 now has only to support the man's whole weight (say W) less P, or W − P. Each piece of the cord that runs from the beam under pulley 1 up to pulley 2 carries $\frac{W-P}{2}$, neglecting the weight of pulley 1; ∴ the Power = $\frac{W-P}{2}$. But the Power is part of the man's weight, represented by P; ∴ $\frac{W-P}{2} = P$, or W − P = 2P, or W = 3P. The man therefore must exert a force = $\frac{1}{3}$ of his own weight to cause equilibrium.

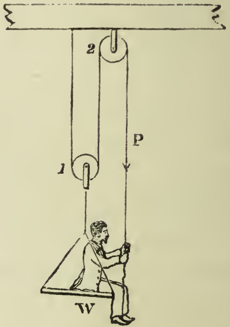


Fig. 76.

Example 40.—What must be a man's weight if, when he is suspended from the lowest of a set of 4 pulleys arranged according to the first system, he can just support himself by pulling vertically downwards at the end of the free cord passing over the fixed pulley, with a force = $15\frac{5}{9}$ lbs.? The line representing the cord from the fixed pulley to the man's hand in Fig. 77 ought to be vertical.

P exerted by the man = $15\frac{5}{9}$ lbs. W = so much of the weight of man as is upheld by pulleys. $15\frac{5}{9}$ transmitted over pulley 1, under moveable pulley 2 to beam.

∴ tension of $15\frac{5}{9}$ in each string supporting pulley 2 = $31\frac{1}{9}$ transmitted to pulley 3.

∴ tension of $31\frac{1}{9}$ in each string supporting pulley 3 = $62\frac{2}{9}$ transmitted to pulley 4.

∴ $62\frac{2}{9}$ in each string supporting pulley 4 = $124\frac{4}{9}$ upheld by pulley 4.

But P exerted by man = part of his own weight;

∴ full weight

$$= W + P = 124\tfrac{4}{9} + 15\tfrac{5}{9} = 140.$$

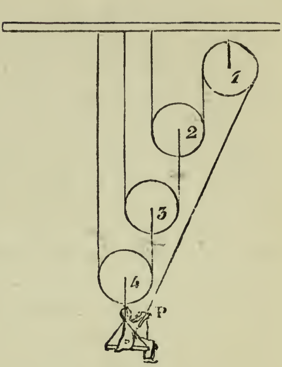


Fig. 77.

Or, let x = weight of man. $W = P \times 2^n$ (where n is the number of moveable pulleys);

$$\therefore \frac{W}{2^3} = 15\tfrac{5}{9};$$

$$\therefore \frac{x - 15\frac{5}{9}}{8} = 15\tfrac{5}{9},$$

$$x - 15\tfrac{5}{9} = 15\tfrac{5}{9} \times 8,$$

$$x = 124\tfrac{4}{9} + 15\tfrac{5}{9}$$

$$= 140.$$

(γ) *With the cords not parallel.*—Hitherto, in all the cases of pulleys that have been considered, the cords have been arranged parallel to each other, and in a vertical direction. And this is clearly the best arrangement to obtain a mechanical advantage. For, consider Fig. 78. In the left-hand pulley the two parts of the cord are parallel and vertical. The tensions in both are working fully together, and W = their sum, *i.e.* the greatest possible resultant.

But in the right-hand pulley the two parts of the cord are not parallel. The tensions in the two are not working fully together. They are to some extent pull-

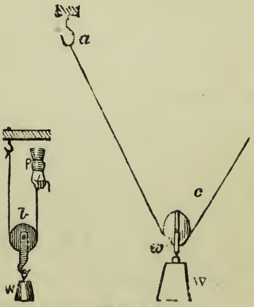


Fig. 78.

ing against each other. W is not = their sum; it is between their sum and their difference.

To find the exact ratio of P to W in such a case as this last, one of two methods may be adopted.

1. Resolution of forces.—Let COO′C′ represent the cord in a case where the two parts are not parallel. Along each part of the cord mark off equal parts OC, O′C′ to represent the tension in that part of the cord. Decompose the forces thus represented each into two forces—one vertical, OA, O′A′; one horizontal, OB, O′B′. The two latter, or the horizontal forces, are equal and opposite. They therefore neutralize one the other, and may be neglected as far as the supporting of the Weight is concerned. But the two vertical forces represented by OA, O′A′ are parallel, and the resultant of them = their sum. Hence W = not the tension along OC + tension along O′C′, but = the sum of the forces represented by OA and O′A′.

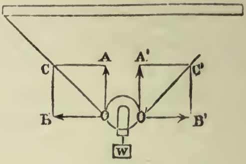


Fig. 79.

2. Triangle of forces.—Three forces are in equilibrium, viz. the Weight acting vertically downwards, the tension along one part of the cord, the tension along the other. A triangle must be obtained, whose sides, taken in order, are severally parallel or perpendicular to the directions of the three forces. The lengths of those sides will be proportional to the magnitudes of the forces.

Example 41.—Find—1. by resolution, 2. by triangle of forces, the Weight that could be sustained by the help of one moveable pulley, when the two parts of the cord make with one another an angle = 120°, and the force applied at the end of one of them = 20 grams.

Let a pulley S be supported by a cord fixed at T, and with a force P acting at the free end. Let the two parts of the cord at the point of support of the pulley make an angle of 120°. It is required to find W, the Weight supported.

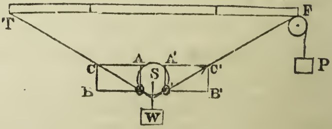


Fig. 80.

1. On OT cut off a part OC. On O′F cut off O′C′ = OC. Decompose OC into forces, one

horizontal represented by OB, one vertical represented by OA. Similarly, decompose O′C′ into forces represented by O′B′, O′A′.

Then OB and O′B′ are equal and opposite, and therefore neutralize each other. OA, O′A′ now represent the upward tensions, and their sum represents the force equal and opposite to W.

Now the $\triangle$ O′A′C′ has at A′O′C′ an $\angle = 60°$, at O′A′C′ a right angle, and $\therefore$ $\angle$ A′C′O′ $= 30°$. In such a $\triangle$, the side opposite 90° is represented by a, that opposite 30° by $\frac{a}{2}$. A′O′ is the side opposite $\angle$ 30°; $\therefore$ A′O′ is represented by $\frac{a}{2}$. Now a represents 20; $\therefore$ $\frac{a}{2}$ represents 10, and $10 \times 2 = 20 = W$.

2. Draw through C a vertical line to represent the direction of W. Along CT cut off a part CB to represent the tension along CT. Draw from B, BA parallel to CD, to meet CA in A. In the $\triangle$BAC thus formed, CA is parallel to and represents W; CB is parallel to and represents tension along CT; BA is parallel to and represents tension along CP; and the $\angle$ BCA is an $\angle$ of 60°; $\because$ CA bisects $\angle$ TCP $= \angle$ of 120°, and BA = BC, as these represent two equal tensions. But if BA = BC, the $\triangle$ABC is isosceles, and $\angle$ BAC $= \angle$ BCA; $\therefore$ $\angle$ BAC is an $\angle$ of 60°, and the remaining $\angle$ ABC also is an $\angle$ of 60°; $\therefore$ the $\triangle$ is equilateral, and P = W.

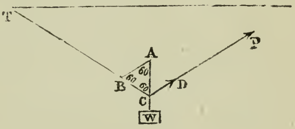


Fig. 81.

Exercises.

1. The first system of pulleys, four pulleys at your disposal, weighing from above downwards $3\frac{1}{2}$, 4, $4\frac{1}{2}$, 5 grams; W = 115 grams. Find P.

2. Same system, five pulleys at your disposal, of equal weight. If when W = 41 grams, P = $5\frac{3}{8}$ grams, find the weight of each pulley. 3

3. The second system of pulleys, with W = 88 grams, 8 pulleys in all at work, and each pulley weighing 2 grams more than the next highest, the highest having weight = 2 grams. Find P.

✓ 4. Second system, six pulleys in use; W = 42 grams; weights of pulleys increase by three in the order in which the cord passes from P over them. If P = $12\frac{1}{2}$, find weights of pulleys.

5. Third system, four pulleys, weighing from below upwards 5, 4, 8, 6 grams. Find W.

6. Same system, three pulleys, weighing each 2 grams; W = $14\frac{1}{2}$. Find P.

CHAPTER XII.

(*e*) THE INCLINED PLANE.

If a heavy body is resting upon a smooth horizontal surface, two forces are in equilibrium. These are the weight of the body, and the resistance of the plane. But when two forces are in equilibrium, they must be equal and opposite one to the other. The first force, the weight of the body, acts vertically downwards ; ∴ the second force, to be exactly opposite, must act vertically upwards. The resistance of the horizontal plane acts, then, vertically upwards, or at right angles to the plane.

If, now, one end of such a horizontal plane is raised whilst the other remains stationary, the plane becomes an inclined plane. Any change in the position of the plane means a like change in the direction of its resistance, as the relation between the direction of the plane and the direction of its resistance to any body upon it remains constant. These two directions are always at right angles. Now, therefore, that the plane is inclined, the two forces, the weight of body upon the plane, and the resistance of the plane, are no longer in equilibrium, as they are not opposite to one another. Therefore the weight begins to slide down the plane if there is no friction, and some third force must be introduced to produce equilibrium. It is very important to keep in mind that the resistance of any smooth, hard surface, whether horizontal or inclined, is at right angles to the surface.

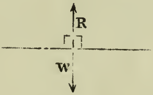


Fig. 82.

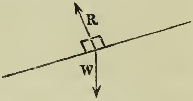


Fig. 83.

Let EDI represent an inclined plane. It is of the form of a right-angled triangle. EI, the hypotenuse of such a triangle, is known as the length of the plane, and will be henceforth denoted by the letter L. DI, the perpendicular of such a triangle, is known as the height, and will be henceforth denoted

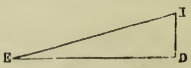


Fig. 84.

by the letter H. ED is the base, which will be represented by the letter B. When you are travelling on a railroad, you often see small posts bearing pieces of wood with figures inscribed thereon, as in Fig. 85. These tell you that you are traversing inclined planes. If the train is going in the direction of the arrow, before it reached the post it was ascending a plane 'rising 3 in 500,' *i.e.* whose height = 3 feet or yards when its length = 500 feet or yards. After the post is passed, the train is descending an inclined plane whose height = 7 when its length is 1250.

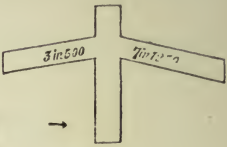


Fig. 85.

(i.) The Inclined Plane when the Power is Horizontal.

If a body rolling down a smooth inclined plane be prevented from doing so by a horizontal force, *i.e.* one parallel to the base of the plane, the conditions obtaining are represented in Fig. 86. Acting upon the ball that is on the plane, are three forces —(α) its own Weight acting vertically downwards, and represented by W; (β) the Resistance of the plane, acting at right angles to the plane, and represented by R; (γ) the Power acting horizontally, represented by P.

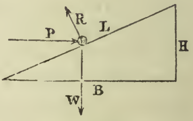


Fig. 86.

These three forces are in equilibrium; we must seek, therefore, for a triangle whose three sides, taken in order, are severally parallel or severally perpendicular to the directions of those three forces. The inclined plane itself is such a triangle.

The direction of W is at right angles to B;

$\therefore$ W is proportional to B

The direction of R is at right angles to L;

$\therefore$ R is proportional to L.

The direction of P is at right angles to H;

$\therefore$ P is proportional to H.

Hence in the inclined Plane, when the Power acts horizontally, it is represented by the height of the plane, the Weight is represented by the base of the plane, and the Resistance by the length.

Example 42.—On an inclined plane whose height is 7 centimetres, and base 3 metres, what power can just support a weight of 1 decagram 8 grams?

In this plane, H = 7 centimetres, B = 300 centimetres, W = 18 grams.

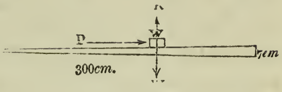


Fig. 87.

The power P acts horizontally, *i.e.* parallel to B.

It is required to find P.

P : W :: H : B.

$$\frac{P}{W} = \frac{H}{B};$$

$$\therefore \frac{P}{18} = \frac{7}{300},$$

$$300P = 7 \times 18,$$

$$50P = 7 \times 3,$$

$$P = \frac{21}{50} \text{ gram.}$$

(ii.) The Inclined Plane when the Power is parallel to the Length of the Plane.

If a body rolling down a smooth inclined plane be prevented from doing so by a force parallel to the length of the plane, the conditions obtaining are represented in Fig. 88. Acting upon the body that is on the plane, are three forces —(α) its own Weight acting vertically downwards, and represented by W; (β) the Resistance of the plane acting at right angles to the plane, and represented by R; (γ) the Power acting parallel to the length of the plane, and represented by P.

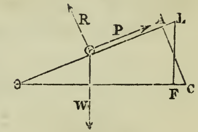


Fig. 88.

These three forces are in equilibrium; we must seek, therefore, for a triangle whose three sides, taken in order, are severally parallel or severally perpendicular to the directions of the three forces. Let us see first if the plane itself will help us.

OL or the length is parallel to P,
LF ,, height ,, W.
But OF ,, base is not parallel to R.
Again, OF ,, base is perpendicular to W
OL ,, length ,, ,, R.
But LF ,, height is not perpendicular to P.

Clearly, then, the inclined plane itself will not serve as our triangle. We must construct another. From OL cut off a part

OA = OF. Extend OF until OC = OL. Join AC. A second triangle OAC is thus constructed, whose sides OA, OC = the sides OF, FL of the first triangle or inclined plane, and whose angle AOC = the angle LOF of the first triangle; ∴ these two triangles, the first OLF and the second OCA, are wholly equal.

Consider the second triangle OCA.

W is perpendicular to OC, and therefore proportional to OC.
P ,, ,, AC, ,, ,, ,, ,, AC.
R ,, ,, OA, ,, ,, ,, ,, OA.

Hence as OC, AC, OA are respectively = OL, LF, OF of the inclined plane, W : P : R as OL : LF : OF, or as length : height : base. Hence in the inclined plane, when the Power acts parallel to the length of the plane, it is represented by the height, the Weight is represented by the length, and the Resistance by the base.

Example 43.—An inclined plane with height = 1 decimetre 4 centimetres when the base = 4 decimetres 8 centimetres has a ball placed on it weighing ½ kilogram. A string is attached to the ball and passes over a small fixed pulley at the summit of the plane. What weight must be hung from the free end of that string to prevent the ball from rolling down the plane, and what will be the pressure on the plane?

Let ABC be an inclined plane, D a ball, upheld by a string that passes over a pulley N, and has a weight P attached to the free end.

It is required to find P, and the pressure on the plane.

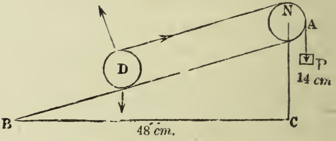


Fig. 89.

In this case the power P is acting parallel to the plane, *i.e* parallel to the length;

$$\therefore P : W :: H : L. \quad \text{Now } L^2 = B^2 + H^2 = 48^2 + 14^2;$$

$$\therefore L = 50.$$

$$\frac{P}{W} = \frac{H}{L}; \therefore \frac{P}{500 \text{ grams}} = \frac{14}{50},$$

$$50P = 14 \times 500,$$

$$5P = 14 \times 50,$$

$$P = 14 \times 10,$$

$$= 140 \text{ grams}.$$

Also R : W :: B : L, and R is equal and opposite to the pressure on the plane.

$$\frac{R}{W}=\frac{B}{L};\ \therefore\ \frac{R}{500 \text{ grams}}=\frac{48}{50},$$

$$\begin{aligned} 50R &= 48 \times 500, \\ R &= 48 \times 10 \\ &= 480 \text{ grams}; \end{aligned}$$

∴ weight attached to free end of string is 140 grams, and pressure on the plane = 480 grams.

EXERCISES.

1. In an inclined plane whose base is a metre and height 3 decimetres, find P if W = 40 grams, and W if P = 40 grams (P parallel to length of plane).

2. What is the pressure on the plane when a horizontal force of 20 grams sustains a weight on a plane rising 3 in 5?

3. The angle of a plane is 60°. If P when horizontal = 9 grams, find W and R.

4. A power of 17 grams works parallel to the length of a plane. If the height and base are respectively $10\sqrt{11}$ and 50 decimetres, find W.

5. A weight is sustained upon an inclined plane by a power parallel to the plane = $\frac{1}{4}$W. If the height of the plane = 2 decimetres, find its length.

6. On an inclined plane whose angle = 30°, a mass is kept up by two forces—(1) of $\sqrt{3}$ grams acting in a horizontal direction,
(2) of 3 " " parallel to the plane.
Find the weight of the mass.

CHAPTER XIII.

(*f*) THE WEDGE.

If two equal inclined planes be placed base to base, with their lengths turned away from each other, an isosceles triangle results. The simple machine thus represented is the wedge. The wedge is therefore a double inclined plane. It is used to force parts of bodies asunder, and familiar examples of it are axes and knives.

In Fig. 90 there is a representation of an isosceles wedge forced into the cleft of a tree, by application of pressure at right angles to its base B. If we call the total pressure exerted by the tree R, we have then three forces in equilibrium. Pressure of the applied force P acting at right angles to the base B. $\frac{R}{2}$ acting at right angles to the side L. $\frac{R}{2}$ acting at right angles to the side L′. As these three forces are in equilibrium, we must look for a triangle whose three sides taken in order are severally parallel or severally perpendicular to the directions of these three forces. The wedge itself is such a triangle.

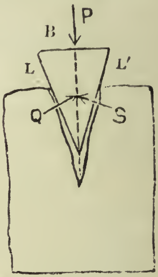


Fig. 90.

P is perpendicular to the base B and therefore proportional to B.
$\frac{R}{2}$,, ,, side L ,, ,, ,, L.
$\frac{R}{2}$,, ,, ,, L′ ,, ,, ,, L′.

But $L = L'$; ∴ P is represented by B, and $\frac{R}{2}$ by L.

Hence in a wedge the force applied at right angles to the base is represented by the base, and the total resistance by twice one of the sides.

Example 44.—What is the vertical height of an isosceles wedge if a pressure of 10 grams applied at right angles to the base balances a total resistance = 1 hectogram, the base being 4 decimetres long.

Let ABC represent a wedge; let the pressure on the base be 10 grams; let the length of the base be 4 decimetres; let the resistance be 1 hectogram.

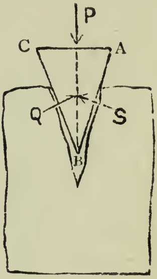


Fig. 91.

It is required to find the vertical height of the wedge.

$$\frac{R}{2} = 50 \text{ grams.}$$

10 grams are represented by a length of 4 decimetres.

1 gram is represented by a length of $\frac{4}{10}$ of a decimetre.

50 grams are represented by a length of $\frac{4}{10} \times 50 = 4 \times 5 = 20$ decimetres.

The wedge has therefore sides of 20 decimetres each.

To find the height PB, draw a vertical line from the apex of the wedge to the centre of the base; in the △ APB thus formed $AB = 20$ decimetres, $AP = 2$ decimetres.

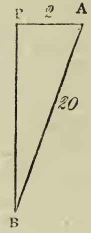


Fig. 92.

Now $AB^2 = AP^2 + PB^2$,
$20^2 = 2^2 + PB^2$,
$400 = 4 + PB^2$,
$400 - 4 = PB^2$,
$\sqrt{396} = PB = 19{\cdot}9$ nearly.

EXERCISES.

1. To the base of an isosceles wedge, whose angle is 60°, is applied a pressure of 1 kilogram. Find the resistance overcome if the vertical height of the wedge is $5\sqrt{3}$ decimetres.

2. What is the length of one side of an isosceles wedge if its base is 2 decimetres, and the ratio of Power to Resistance is $\frac{1}{2}$ to 10?

CHAPTER XIV.

(*g*) THE SCREW.

If the student will cut out in paper a right-angled triangle, and wind the paper triangle round a cylindrical ruler, he will notice that the hypotenuse describes a spiral line round the ruler, just such a spiral line as is observed in an ordinary screw. In fact, the screw is an inclined plane wound round a central pillar or modiolus.

Take the paper right-angled triangle of such a size as to admit of being wrapped exactly once round the ruler, and observe what becomes of the three parts height, base, length of the inclined plane. The length, we have seen, forms the spiral thread of the screw. The base describes the circumference of the central pillar. The height corresponds with the vertical distance between any two successive turns of the thread.

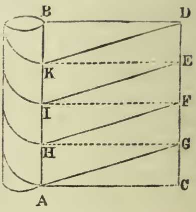


Fig. 93.

The screw, therefore, is a modification of the inclined plane, where L becomes the spiral thread, B the circumference of the screw, H the distance between the threads. The Power is usually applied in this simple machine by means of a horizontal lever working perpendicular to the axis of the screw, and with fulcrum in that axis. Our inclined plane, therefore, is worked by a Power parallel to the base. But by page 90,

$$P : W :: H : B,$$

and in the screw this becomes

P : W :: distance between threads : circumference.

Hence, in the screw the Power is represented by or is proportional to the distance between the threads, and the Weight is represented by or is proportional to the circumference of the screw.

When a lever is used, the circumference of the screw practically becomes the circumference traced out by the end of the lever as it turns round.

Example 45.—With a screw having 4 threads to the centimetre, and a radius $= \frac{1}{2}$ decimetre, find the Power necessary to support a Weight $= 44$ grams.

In the screw P : W :: H : B.

Let $W = 44$ grams, $H = \frac{1}{4}$ cm. B, the circumference, must be found from the radius. Circumference $= 2\pi r = \frac{44}{7} \times 5$ cm. It is required to find P.

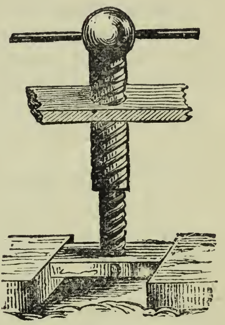


Fig. 94.

$$\frac{P}{44} = \frac{\frac{1}{4}}{\frac{44 \times 5}{7}},$$

$$\frac{P}{44} = \frac{1}{4} \times \frac{7}{44 \times 5},$$

$$P = \frac{7}{4 \times 5},$$

$$P = \frac{7}{20} \text{ gr.}$$

Example 46.—What must be the length of a lever which, when made to work a screw with 7 threads to the centimetre, gives a mechanical advantage $= 1320$?

Let $x =$ the circumference traced by the extremity of lever; let $H = \frac{1}{7}$ cm.; let the mechanical advantage $= 1320$.

It is required to find the length of the lever.

$$P : W :: H : x,$$

$$\frac{1}{1320} = \frac{1}{7x}.$$

But if $x =$ circumference, $x = 2\pi r = \frac{44}{7} \times r$;

$$\therefore \frac{1}{1320} = \frac{1}{7 \times \frac{44}{7} \times r},$$

$$7 \times \frac{44}{7} \times r = 1320,$$

$$44r = 1320,$$

$$r = \frac{1320}{44} = 30.$$

Exercises.

1. A screw with 10 threads to the decimetre, and a radius = 14 millimetres, has a Power of 2 grams applied to it. Find W.

2. The pressure on a screw with 2 threads to every 3 centimetres, and a diameter = 7 millimetres, is 33 grams. What Power will maintain equilibrium?

3. How many threads will there be in 1 decimetre length of a screw if a Weight is supported by a Power = $\frac{1}{7}$W, radius $11\frac{3}{22}$ millimetre?

4. Find the radius of a screw in which W = 28 grams when P = 2 grams, the distance between the threads being 1 centimetre.

5. A lever of length $1\frac{1}{2}$ metre works a screw with 11 threads to the decimetre. Find W when P = 77 milligrams.

6. What is the length of a lever which, working a screw, the distance between whose threads is 2 millimetres, gives a mechanical advantage of 70?

CHAPTER XV.

(*h*) COMBINATIONS OF MACHINES.

The various simple machines have now been considered, and it may be well briefly to sum up the relations between P and W in each.

Modifications of Lever	**Lever**		$P : W :: $ W's arm : P's arm.
	Wheel and Axle		$P : W :: $ radius of axle : radius of wheel.
	Toothed Wheels		$P : W :: $ number of teeth in P's wheel : number in W's.
	Pulleys	Fixed Pulley	$P = W$.
		Moveable Pulley	$P : W :: 1 : 2$.
		First System	$P : W :: 1 : 2^n$. n=number of moveable pulleys.
		Second ,,	$P : W :: 1 : 2n$. n=number of moveable pulleys.
		Third ,,	$P : W :: 1 : (2^n - 1)$. n=number of pulleys.
Modifications of Inclined Plane	**Inclined Plane**	P horizontal	$P : W :: H : B$.
		P parallel to plane	$P : W :: H : L$.
	Wedge		$P : W :: B :$ sum of two sides.
	Screw		$P : W :: $ d.b.t. : circumf.

These seven simple machines may be grouped into very complex relationships. It will be obvious that the majority of these cannot be studied here. (i.) But in dealing with the screw it was pointed out that, as a rule, the lever was employed with the screw.

(ii.) Again the *endless screw* is a combination of the screw and the wheel and axle.

In this combination the screw is turned by a lever called the winch, and its threads press against the teeth of the wheel. The Power acts upon the end of the winch or lever; the Weight is attached to the circumference of the axle. Consider first the winch and screw only.

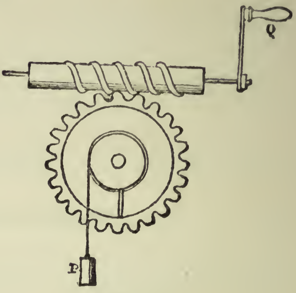


Fig. 95.

P : W :: d. b. t. : circumference described by end of winch;

$$\therefore\ W = \frac{P \times \text{circ.}}{\text{d. b. t.}}.$$

Consider next the wheel and axle.

P : W :: radius of axle : radius of wheel;

$$\therefore\ W = \frac{P \times \text{radius of wheel}}{\text{radius of axle}}.$$

But with the wheel and axle, the Power = the Weight in the screw; substituting for P in the second equation the value of W in the first, we have

$$\text{Final } W = \frac{\text{original } P \times \text{circ.} \times \text{radius of wheel}}{\text{d. b. t.} \times \text{radius of axle}}.$$

Example 47.—An endless screw is worked by a winch one metre long. The distance between the threads = one centimetre. The diameters of the wheel and the axle are respectively one metre and five centimetres. If P = 7 grams, find W.

Let P = 7 grams, length of winch = 100 cm., distance between threads = 1 centimetre. It is required to find W.

P : W :: d. b. t. : circumference.

$$\text{Circumference} = 2\pi r$$

$$= \frac{44}{7} \times 100.$$

$$W = \frac{7 \times \frac{44}{7} \times 100}{1} = 44 \times 100.$$

Let diameter of wheel = 100 centimetres, that of axle = 5 centimetres.

P : W :: radius of axle : radius of wheel.

$$W = \frac{P \times 50}{2\frac{1}{2}};$$

$$\therefore \text{ final } W = \frac{44 \times 100 \times 50}{1 \times 2\frac{1}{2}} = \frac{44 \times 100 \times 50 \times 2}{5}$$

$$= 44 \times 100 \times 10 \times 2 = 88000 \text{ grams} = 88 \text{ kilog.}$$

(iii.) A *crane* is another combination of wheels and axles.

In working examples upon all combinations of machines, notice that, as in the endless screw, the Weight of the first machine becomes the Power of the second, and so on.

If the meaning of the convenient phrase mechanical advantage be kept in mind, much trouble can be frequently saved in dealing with a combination of machines. Mechanical advantage $= \frac{W}{P}$, *i.e.* the mechanical advantage of any machine is represented by an abstract number that is equal to the quotient when the Weight is divided by the Power.

Suppose we have a number of machines working in combination.

Let M, M′, M″, etc. = their respective mechanical advantages.

Let P and W, W and W′, W′ and W″, etc. = their respective Powers and Weights.

Consider the first machine, $M = \frac{W}{P}$; $\therefore W = MP$.

„ „ second „ $M' = \frac{W'}{W}$; $\therefore W' = M'W$.

„ „ third „ $M'' = \frac{W''}{W'}$; $\therefore W'' = M''W'$.

Multiplying together the second trio of equations, we have

$$WW'W'' = MM'M''PWW'.$$

Dividing this equation by WW′;

$$\therefore W'' = MM'M''P;$$

$$\therefore \frac{W''}{P} = MM'M'',$$

or the final Weight ÷ the original Power = the product of the successive mechanical advantages, *i.e.* the mechanical advantage of any combination of machines = the product of the separate mechanical advantages of the various machines entering into the combination.

Example 48.—A lever of the second kind whose length = 15 decimetres, and whose Weight is applied at a point $\frac{1}{2}$ metre from the fulcrum, works a second lever of length 8 decimetres that turns a screw, the distance between whose threads is 2 centimetres. The screw is an endless one, and the circumference of the wheel worked by it is 12 decimetres, while the diameter of the axle of that wheel is 4 centimetres. Find the mechanical advantage of the combination.

First mechanical advantage:

$$W : P :: \text{P's arm} : \text{W's arm}.$$

$$\frac{W}{P} = \frac{150 \text{ cm.}}{50 \text{ cm.}} = 3.$$

Second mechanical advantage:

$$W' : P' :: \text{circumference traced by lever} : \text{distance between threads}.$$

$$\frac{W'}{P'} = \frac{\frac{44}{7} \times 80}{2}$$

$$= \frac{44 \times 80}{7 \times 2}$$

$$= \frac{22 \times 80}{7} = \frac{1760}{7}.$$

Third mechanical advantage:

$$\text{Circumference} = 2\pi r,$$

$$120 = \frac{44}{7} r,$$

$$r = \frac{120}{\frac{44}{7}} = \frac{120 \times 7}{44}.$$

$$W'' : P'' :: \text{radius of wheel} : \text{radius of axle}.$$

$$\frac{W''}{P''} = \frac{120 \times 7}{44} \div 2 = \frac{120 \times 7}{88} = \frac{15 \times 7}{11}.$$

$$\text{Total mechanical advantage } \therefore = 3 \times \frac{1760}{7} \times \frac{15 \times 7}{11}$$

$$= 3 \times 160 \times 15$$

$$= 7200$$

EXERCISES.

1. If the mechanical advantage of a lever of the first kind is represented by 12, what is the length of the lever if the distance between the fulcrum and the point of suspension of the weight is 3 centimetres?

2. What must be the circumference of a wheel whose axle has radius 2 centimetres when the mechanical advantage is 21?

3. An inclined plane has ∠60°, and the Power works hori-

zontally; another has ∠30°, and the Power works parallel to plane. Compare their mechanical advantages.

4. What is the diameter of a screw with mechanical advantage $25\frac{1}{7}$ if it has 4 threads to the centimetre?

5. A lever of the first kind whose arms are 12 and 2 works one of the second kind, whose length is 1 metre and the distance between Power and Weight 6 decimetres. The second lever works a screw with 3 threads to the millimetre and radius 7 millimetres. Find the mechanical advantage.

CHAPTER XVI.

THE SIMPLE MACHINES AND THE PRINCIPLE OF WORK.

(i.) WORK.

If a body moves through a certain space against the action of some force, work is done. The measure of the work done depends upon two things—(α) the opposing resistance, (β) the distance moved through by the working body against that resistance. As the amount of work varies directly as both these, it varies as their product. Therefore work is represented by the product of the resistance overcome, and the *effective* distance through which the working body moves. If, *e.g.*, a man push a mass of 40 grams along a smooth horizontal plane for a distance of 3 metres, $40 \times 3 = 120$ is the abstract number that represents the work done. If he push a kilogram along the same plane 2 decimetres, 1000 (grams) $\times \frac{1}{5}$ (metre) $= 200$ is the abstract number that represents the work done.

Usually the units adopted are a kilogram and a metre. To have moved a mass of one kilogram through one metre (1×1) is to have done a unit of work. Taking the 'kilogrammetre' as the unit in the two cases just given, the amount of work done in work units =

in the first case, $\frac{40}{1000} \times 3 = \frac{120}{1000} = \frac{12}{100} = \frac{3}{25}$;
in the second case, $1 \times \frac{1}{5} = \frac{1}{5}$.

Notice the phrase *effective* distance used above. Effective distance means the distance moved through in a direction exactly opposite to the direction of the resistance. In the cases cited above there is no difficulty. But in the case of an inclined plane care must be exercised.

If a mass of 2 kilograms were pushed up an inclined plane rising 3 in 10, and the *length* of the plane were exactly 10 metres, the work done is *not* 2 × 10 or 20. For work is represented by the product of the resistance overcome (mass of 2 kilograms), and the *effective* distance through which the working or pushing body moves. Now effective distance means the distance moved through in a direction exactly opposite to the direction of the resistance overcome. The mass of 2 kilograms is tending vertically downwards towards the earth. The 'direction exactly opposite' is vertically upwards. The effective distance is the distance accomplished in a vertically upward direction, and that in the case in point is not 10 but 3;

∴ work done = 2 × 3 or 6 units.

(ii.) ENERGY.

Energy is the capacity for doing work. It is believed that energy can no more be created or destroyed than can matter. The total amount of capacity for work in the universe to-day = the total amount of capacity for work in the universe yesterday, or on any other of the countless yesterdays. Hence in any of the machines, when the Power does a certain amount of work, the Weight does the same amount of work. Neglecting friction and any heat or electrical work that may be effected by the way, the work done by the Power in any machine = the work done by the Weight. Remembering that the measure of work done is the product of the mass moved, and the effective distance through which it is moved, we proceed to apply this principle to the simple machines, one after the other, to see if we obtain results consonant with those recorded upon p. 99.

(α) The Lever.

If VL represent a lever of the first kind, OV one of the second, and OL one of the third kind, and V is the point of application of the Power, L that of the Weight, and O the fulcrum, when V has moved to V′, L is raised to L′;

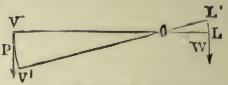


Fig. 96.

$\therefore$ by principle of work, $P \times VV' = W \times LL'$.

But the two triangles VOV′, LOL′, are similar, as

$\angle VOV' = \angle LOL'$, and $\angle OVV' = \angle OLL'$;

$$\therefore \frac{VV'}{LL'} = \frac{OV}{OL}.$$

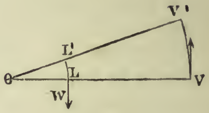


Fig. 97.

Hence in our equation we can substitute for VV′ and LL′, OV and OL;

$$\therefore P \times OV = W \times OL, \text{ or}$$

the Power × its arm = the Weight × its arm.

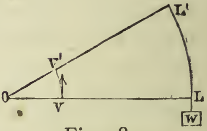


Fig. 98.

(β) The Wheel and Axle.

1. Simple.—The work done by P during one complete revolution of the wheel = P × circumference of wheel. The work done by W in the same time = W × circumference of the axle. Let C denote the wheel's circumference, c that of the axle;

$$\therefore P \times C = W \times c.$$

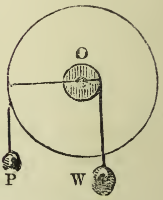


Fig. 99.

But the radii of circles are proportional to their circumferences. Hence we can substitute in our equation R (radius of wheel) for C, and r (radius of axle) for c;

$$\therefore P \times R = W \times r,$$

or the Power × the radius of the wheel

= Weight × ,, ,, axle.

If only a fraction of the entire revolution is performed, the *relations* between the spaces passed through by P and W remain the same.

2. COMPOUND.—During one complete revolution, P's work = P × C.

The Weight is in the same time raised through a height = the circumference of the larger axle (say c), and lowered through a depth = the circumference of the smaller (say c'). On the whole then it is raised through $c - c'$. Its work = $W \times (c - c')$; $\therefore P \times C = W \times (c - c')$. For C we may as above substitute R, and for $(c - c')$ the difference between the radii of the two axles, say $(r - r')$; $\therefore P \times R = W(r - r')$, or the Power × the radius of the wheel = the Weight × the difference between the radii of the two axles.

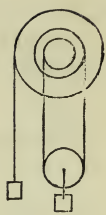


Fig. 100.

(γ) Toothed Wheels.

With one complete revolution of the left-hand wheel, P descends a distance equal to the circumference of the axle that carries it (say c). But one revolution of the left-hand wheel does not mean a revolution of the right-hand and larger one, but only a fraction of a revolution. If, *e.g.*, the left-hand wheel were one-half the size of the right-hand one, a complete revolution of the former would cause one-half a revolution of the latter, and W would only be raised $\frac{1}{2}$ of c. (The two *axles* are equal.) Or more generally, if r and R represent the radii of the two wheels, when P descends a distance $= c$, W ascends a distance $= \frac{r}{R}$ of c;

$$\therefore P \times c = \frac{Wr}{R} \times c;$$

$$\therefore P = \frac{Wr}{R}; \therefore PR = Wr.$$

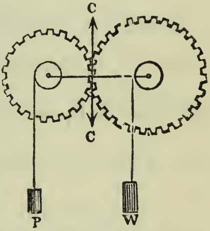


Fig. 101.

But radii of circles are proportional to circumferences, and in toothed wheels with equal teeth and equal intervals, circumferences are proportional to the number of teeth cut in them;

$\therefore$ P × number of teeth in large wheel
= W × number of teeth in small wheel.

(δ) Pulleys.

1. Fixed.—$P = W$, and the distances they each move through are equal. Let it be d.

$$P \times d = W \times d.$$

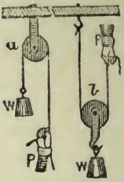


Fig. 102.

2. Moveable.—If P ascends a distance $= d$, each part of the cord is only shortened to an extent $= \frac{d}{2}$;

$$\therefore\ P \times d = \frac{Wd}{2};$$

$$\therefore\ P = \frac{W}{2}.$$

3. First System.—If P descend a distance d, each part of the cord passing under the top moveable pulley is shortened $\frac{d}{2}$.

If the top moveable pulley is raised a distance $= \frac{d}{2}$, it shortens each part of the cord attached to it, and passing under the second moveable pulley to an extent $= \frac{d}{4}$.

In like fashion, the next cord passing under the third moveable pulley is shortened $\frac{d}{8}$, and if there are n moveable pulleys, when P descends through a distance $= d$, W is raised through a distance $= \frac{d}{2^n}$;

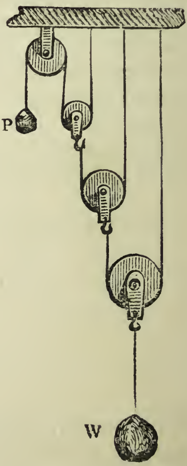


Fig. 103.

$$\therefore\ P \times d = \frac{Wd}{2^n};$$

$$\therefore\ P = \frac{W}{2^n}.$$

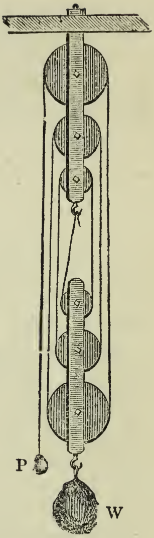


Fig. 104.

4. Second System.—If P descend through a distance $= d$, and there are n pulleys in the lower block, and therefore $2n$ pieces of cord supporting the weight, each piece of cord must be shortened to an extent $= \frac{d}{2n}$;

$$\therefore \ P \times d = \frac{Wd}{2n};$$

$$\therefore \ P = \frac{W}{2n}.$$

5. Third System.—It will be found more easy in this case to start with the work done by the Weight. Suppose the Weight is raised a distance $= d$. Each of the cords passing from the weight over a pulley is shortened d. The Power would therefore descend through a distance $= d$ from the shortening of the right-hand cord *alone*. But as the second cord, counting from the right, is also shortened to an extent $= d$, pulley B is lowered a distance $= d$, and the cord over it unwinds to an extent $= 2d$; $\therefore$ the Power is lowered through a distance $= d$ (from shortening of first cord) $+ 2d$ (from shortening of second).

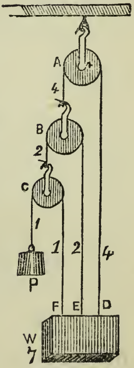


Fig. 105.

But again, the third cord, counting from the right, is also shortened through a distance $= d$; $\therefore$ pulley C is lowered through a distance $= d$ on account of the shortening of the third cord, and the cord over pulley C unwinds to an extent $= 2d$. Thus, pulley C is lowered $2d$, and the cord over it unwinds to an extent $= 4d$, and lowers the Power through a distance $= 4d$.

Thus the Power is lowered through—

1. d from shortening of first cord,
2. $2d$ from shortening of second cord,
3. $4d$ from shortening of third cord;

or d with one pulley, $3d$ with two, $7d$ with three;

or $(2^n - 1)d$ with n pulleys;

$$\therefore\ P(2^n - 1)d = Wd; \quad \therefore\ P(2^n - 1) = W.$$

(ϵ) Inclined Plane.

1. With P horizontal.—If a weight W is at O, the lowest point in an inclined plane, and is pushed by a Power P up to O′, the highest point in the plane, P moves through a distance = the base of the plane B, and W is raised through an *effective* distance = H;

$$\therefore\ P \times B = W \times H; \ \therefore\ P : W :: H : B.$$

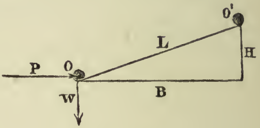


Fig. 106.

2. With P parallel to Length of Plane.—If a Weight W is at the lowest point O in an inclined plane, and is pushed by a Power P up to O′, the highest point in the plane, P moves through a distance = the length of the plane = L, and W is raised through an effective distance = H.

$$\therefore\ P \times L = W \times H; \ \therefore\ P : W :: H : L.$$

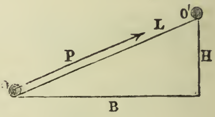


Fig. 107.

(ζ) Wedge.

Before the tree was cleft, the point T was where the point D is now. A Power P has therefore worked through a distance = DT, and P × DT is the measure of its work.

From D draw two lines perpendicular to the sides of the wedge. The points H and A were originally together; ∴ DH + DA represents the distance through which the Weight or resistance has been moved, and one of them, say DA, represents the distance through which $\frac{W}{2}$ has been moved.

$$\therefore\ P \times DT = \frac{W}{2} \times DA. \quad . \quad . \qquad \text{(i)}$$

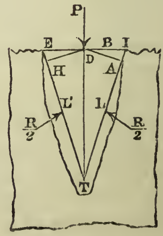


Fig. 108.

Now in triangles DAT, DET, $\angle$ DAT = $\angle$ TDE (right angles), $\angle$ DTA = $\angle$ DTE (angle at T bisected by DT);

$\therefore$ triangles DAT, DET are similar;

$\therefore$ DT : DA :: ET : DE;

Therefore, substituting in equation (i.),

$$P \times ET = \frac{W}{2} \times DE,$$

$$\frac{P}{\frac{W}{2}} = \frac{DE}{ET} \text{ or } \frac{P}{W} = \frac{DE}{2ET},$$

or P : W :: DE : 2ET,

i.e. Power is to whole resistance as base of wedge is to sum of its sides.

(η) Screw.

When the Power has described a complete circle, the Weight has been raised through a height = the distance between two consecutive threads;

$\therefore$ P × circumference of screw = W × distance between threads;

$\therefore$ P : W :: distance between threads : circumference.

The above principle that has now been applied to each of the simple machines, is sometimes known as the principle of virtual velocities, or badly *vis viva.* It is then generally stated thus :

W : P :: space moved through by P : space moved through by W.

Exercises.

These are to be solved by equating work done.

1. With 4 moveable pulleys arranged upon the first system, a Power of 3 grams descends 4 decimetres. Find the Weight and the distance through which it is raised. The pulleys are weightless.

2. If a wheel and axle have diameters respectively of 20 and $\frac{1}{2}$ decimetre, and a Weight of 40 grams is raised through 1 metre, find the Power and the distance through which it falls.

3. A screw having 10 threads to the decimetre is worked by a lever 1 metre long. If the Power is 2 grams, what is the Weight?

4. If with three turns of a lever that works a screw a weight of 30 grams is raised $\frac{1}{9}$ centimetre, find the Power, length of lever $176\frac{76}{99}$ cm.

DIVISION II.

STATICS OF FLUIDS.

CHAPTER XVII.

I. GENERAL PRINCIPLES.

Hitherto we have had under consideration solid bodies in a condition of equilibrium. It is now necessary to study fluids (liquids and gases) in a condition of equilibrium.

A. Solids, Liquids, Gases.

If you take hold of a piece of wood and try to lift one corner of it, you find that you cannot move that corner alone. The parts of the wood adjacent to the seized corner move with it, and unless there be some hinge or like arrangement in the piece of wood, you will find that to raise the corner to any considerable height, it is necessary to move the whole of the wood. Such bodies as this, to move a few of the particles whereof involves the movement of the adjacent particles,—bodies, in short, whose particles do not move easily one over the other, are called *solids*. If you now try to take hold of a piece of water, you find no difficulty in separating a small portion of it from the main body. Such bodies as this, whose atoms can be moved easily one over the other, are called *fluids*. Of fluids there are plainly two kinds, illustrated respectively by water and air. Let us see how these differ. Place an open jar full of water in a room: the water, if untouched, will remain in the jar for a lengthy period, and by no means pass abroad at once into every part of the room. But place an open jar full of chlorine in a room: the chlorine will not remain in the jar for a lengthy period; it at once passes abroad or diffuses into every part of the room. The water is an example of a liquid; air and chlorine are examples of gases.

Solids, then, are bodies whose particles cannot be made to move easily over each other. *Fluids* are bodies whose particles can be made to move one over the other upon the application of very slight force. Of these fluids there are *liquids* whose particles do not rapidly diffuse throughout the whole of any containing space, and *gases* whose particles do rapidly diffuse throughout the whole of any containing space. The particles of gases, in short, seem to repel each other.

We may consider these three kinds of bodies from another point of view. It has been already shown that there is an attraction between particles of like nature. This attraction of similar particles, an attraction only coming into play when the particles are close together, is called cohesion. There is also a force whose most frequent consequence is the separation of particles. That is the force of heat. Heat expands bodies, *i.e.* separates their particles farther and farther asunder. It is therefore evident that cohesion and heat are opposed one to the other. The former tries to keep particles close together; the latter tries to separate them. For these two forces, all nature is the battle-ground. In some cases the victory is won by cohesion; a solid body results. In others, heat gains the day; a gas results. In yet others, what may be called a drawn battle is seen, and a liquid results. So that in solids cohesion is the stronger, and the particles are closely bound one to the other; in gases heat is the stronger, and apparent repulsion of the particles from one another is observed; in liquids, neither cohesion nor repulsion is markedly the more powerful.

B. Principles affecting Fluids generally.

Before considering the statics of each kind of fluid separately, it will be well to master thoroughly two great principles in respect to fluids generally, *i.e.* in respect to liquids *and* gases. These principles are unlike any hitherto encountered in our study of solids, and are of extreme importance.

(1) Suppose a solid block of wood. At one part of its upper surface let a square hole exactly one square centimetre in area be made. Into this let a solid piston be fitted, and a pressure = 1 gram applied to the piston. This pressure will be transmitted in a direct line from the impressed piston, through the solid wood, to that square centimetre in its lower surface which is exactly opposite the piston above. No other square centimetres at all in the whole surface of the solid block of wood will be affected by the gram pressure on the piston.

Now, substitute for the block a box of the same size and shape full of water or of air, with a similar aperture in its upper

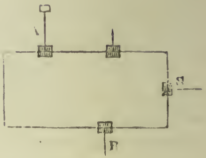


Fig. 109.

surface, and a like piston weighted with one gram. But in addition, let there be other holes, each one square centimetre in area, cut in the upper surface of the box, in the sides, in the base. It is evident that if pistons are fitted in all these apertures, those pistons which are in the sides and the base of the box full of fluid, as S and B, Fig. 109, must be first pressed in with a certain number of grams to overcome the mere pressure outwards of the water before any gram weight is put on A, the top piston. When these side and base pistons have been thus fixed in their places, let the one gram weight be applied to piston A. It will be found that *all* the other pistons having the same area as A must now be pressed in with a force = 1 gram, and if there be a piston whose area is twice that of A, it must be pressed in with a force = 2 grams. No matter how many pistons of one square centimetre in area are used, every one of them must be pressed in with a force = 1 gram to keep it in place if a gram weight is placed on A. Hence it is evident that a pressure = 1 gram is transmitted to every square centimetre in the area of the box. Nothing akin to this occurred in connection with solid bodies, but with all fluids this principle holds :—*If a fluid be contained in a closed vessel, and upon any given area of the surface of that fluid a pressure be exerted, to every portion of the surface of that fluid equal in area to the given area the same pressure is transmitted.* In brief, fluids transmit pressure equally in all directions.

Example 49.—A box 4 decimetres long, 3 broad, and 2 high, has a piston fitting into an aperture that occupies exactly $\frac{1}{4}$ of the top of the box. Calculate the transmitted pressure on one square centimetre in one of the sides, and on the whole of the base, if the piston weighs 100 grams.

Given a box 40 centimetres long, 30 broad, and 20 high, with a piston of area $\frac{1}{4}$ top of box, and of weight 100 grams, it is required to find the transmitted pressure on 1 square centimetre in a side, and on the whole of the base.

The pressure transmitted is the pressure exerted by the weight of the piston. This weight = 100 grams. Area of the piston = $\frac{1}{4}$ of the top of the box. Area of top of box = 40 cm. × 30 cm. = 1200 sq. cm.

$\frac{1}{4} \times 1200 = 300$ sq. cm. = area of piston ;

∴ each square centimetre of the piston exerts a pressure of $\frac{100}{300}$, or $\frac{1}{3}$ gram, and this pressure is transmitted to each square centimetre of surface.

Pressure on 1 sq. cm. of side is ∴ $\frac{1}{3}$ gram. Area of base of box = 40 cm. × 30 = 1200 sq. cm. Pressure on whole base therefore = $\frac{1}{3} \times 1200 = 400$ grams.

(2) In the study of the pistons working in the box, during our investigation of principle (1), it was observed that the pistons fitting into the top of the vessel need no pressure to be exerted upon them to maintain them in place until pressure is exerted upon some special one of them. Those pistons, however, that are fitted into the sides or base, must be 'fixed in their places' ere any such pressure begins to act. And the force requisite to 'fix' them is rendered necessary by the mere pressure due to the weight of the water ere that water *transmits* any pressure at all. Any fluid exerts a certain force, due to its weight, upon any surface immersed in it. It will be necessary to determine the exact amount of this pressure due to fluid weight, and not due to transmission under varying circumstances.

This weight pressure will depend upon three things.—(*a*) The heavier the fluid in which the given area is immersed, the greater will be the pressure upon the area; hence that pressure will vary with the specific or relative weight of the fluid. (*b*) The greater the depth to which the area is sunk in the fluid, the greater will be the pressure upon it; hence the pressure will vary as the depth of immersion. (*c*) The larger the area that is subject to fluid pressure, the greater will be the pressure upon it; hence that pressure will vary as the area of the surface immersed.

(*a*) In considering the relative weight of the fluid, we must partially anticipate a succeeding chapter. Water is taken as the standard of reference for relative weights with liquids; hydrogen is taken as the standard of reference for relative weights with gases. It must be remembered that one cubic centimetre of distilled water, under certain conditions of temperature and pressure, weighs exactly one gram. (For the aid of those who may encounter people sufficiently barbaric to use English weights and measures, it may be mentioned that a cubic foot of water weighs 62·33 lbs., or nearly 1000 ozs., so that the weight of one cubic inch of water may be taken as $\frac{1000}{1728}$ oz.) The weight of one cubic centimetre of any other liquid whose weight in reference to water is given, can easily be calculated if this fact be borne in mind. Again, 11·2 litres of hydrogen gas weigh, under certain conditions of temperature and pressure, one gram. The weight of known volumes of any gas whose weight in relation to hydrogen is known, can thus be calculated.

(*b*) The depth of the area below the surface of the fluid is easily dealt with when the immersed area is horizontal, as in the case of the bottom of a vessel containing fluid. But when the area is inclined at an angle to the horizon or is vertical, the question arises as to how the depth of such a surface is to be calculated. The answer is: Take the mean or average depth of the immersed area. This mean or average depth may be taken as the depth of the centre of gravity of the immersed surface below the surface of the fluid.

(*c*) The area may be found by any suitable one of the ordinary methods for finding areas of surfaces.

If the surface immersed is a rectangle, the area = product of adjacent sides.

If the surface immersed is a square, the area = square of one side.

If the surface immersed is a triangle, the area = half base × height.

If the surface immersed is a circle, the area $= \pi r^2$.

If the surface immersed is that of a sphere, the area $= 4\pi r^2$.

To sum up. The pressure of any fluid upon a surface immersed in that fluid = the weight of a column of the fluid of area = that of the surface immersed and of height = the vertical distance between the surface of the fluid and the centre of gravity of the immersed surface.

Example 50.—A tank 5 metres long, 2 metres broad, and 9 decimetres in depth is filled to within one decimetre of the top with water. Calculate the pressure upon the base and upon each of its sides.

Given a tank 500 centimetres long, 200 broad, and 90 deep, filled to a depth of 80 centimetres with water.

It is required to find the pressure on the base, and on each of the sides.

The mass of water dealt with has a length of 500 centimetres, a breadth of 200, and a depth of 80. The whole of this presses on the base, and each cubic centimetre weighs 1 gram.

Cubical contents = 500 × 200 × 80 = 8,000,000 c.c. Weight = 8,000,000 grams.

On each side presses a column whose base = immersed area of side, and whose height = depth of centre of gravity beneath surface of water.

Immersed area of sides bounded by breadth and depth $= 200 \times 80 = 16,000$ sq. cm.

Depth of centre of gravity below surface of immersed part of fluid $= 40$ cm.

Cubical contents of column $= 16,000 \times 40 = 640,000$ c.c. Weight $= 640,000$ grams.

Immersed area of sides bounded by length and depth $= 500 \times 80 = 40,000$ sq. cm.

Depth of centre of gravity below surface $= 40$ cm.

Cubical contents of column $= 40,000 \times 40 = 1,600,000$ c.c. Weight $= 1,600,000$ grams.

Example 51.—A rectangular vessel $\frac{1}{2}$ metre in depth is filled with a fluid ten times as heavy as water. What are the length and breadth if the former is twice the latter, and the pressure on the long side is 4 kilograms? Also find the pressure on the short side.

Given a rectangular vessel 50 centimetres in depth, with breadth $= x$, length $= 2x$, filled with a fluid ten times as heavy as water, which exerts a pressure on the long side $= 4000$ grams.

It is required to find the length, breadth, and pressure on the short side.

If the fluid were water, 1 centimetre would measure 1 gram; the fluid being ten times as heavy as water, 1 gram measures $\frac{1}{10}$ centimetre; and the cubical contents of a mass weighing 4000 grams $= \frac{1}{10}$ of the cubical contents of a mass of water weighing 4000 grams.

Centre of gravity of side is at a depth of 25 centimetres.

Pressure on side = weight of column whose base is the area of the side, and whose height is depth of centre of gravity below surface of fluid.

Pressure on side = area × depth of centre of gravity × weight of unit of fluid.

$$4000 \text{ grams} = (50 \times 2x \times 25) \text{ c.c. if contents were water.}$$

$$4000 \text{ grams} = \frac{50 \times 2x \times 25}{10} \text{ c.c. with given fluid ten times}$$

as heavy as water.

$$\begin{aligned} 4000 \text{ grams} &= 5 \times 2x \times 25 \text{ c.c.} \\ &= 250x. \end{aligned}$$

$$\frac{4000}{250} = x.$$

$16 = x$, and the breadth $= 16$ cm. and length $= 32$ cm.

Area of short side $= 50x = 50 \times 16$. Let x *now* equal pressure.

$$x = \frac{50 \times 16 \times 25}{10},$$
$$10x = 20{,}000,$$
$$x = 2000 \text{ grams.}$$

Example 52.—A cylindrical vessel filled with a fluid one-half as heavy as water has a diameter of 2 decimetres 8 centimetres. What is its height if the pressure on the base is 44 kilograms?

Given a vessel whose radius $= 14$ centimetres, whose height $= x$, filled with a fluid $\frac{1}{2}$ as heavy as water, and the pressure on the base $= 44{,}000$ grams.

It is required to find x.

If the vessel were full of water, the pressure on the base in grams would = the cubical contents of the liquid column in centimetres.

But the fluid is $\frac{1}{2}$ as heavy as water; ∴ 2 centimetres are required to exert a pressure of 1 gram; ∴ the cubical contents of the column must be twice as great as if the fluid were water. If the area of the base be the same, the column must ∴ be twice as high.

$$r = 14 \text{ centimetres, area} = \pi r^2$$
$$= \frac{22}{7} \times 14 \times 14$$
$$= 22 \times 2 \times 14.$$

Cubical contents weighing 44,000 grams = column of water containing 44,000 cubic centimetres.

∴ cubical contents weighing 44,000 grams = column of liquid $\frac{1}{2}$ as heavy as water containing 88,000 centimetres.

Area of base × height = cubical contents;

$$\therefore 22 \times 2 \times 14 \times x = 88{,}000 \text{ centimetres,}$$
$$14x = 2000,$$
$$x = \frac{2000}{14} = \frac{1000}{7} = 142\tfrac{6}{7} \text{ cm.}$$

After this consideration of the two great principles affecting fluids generally, we turn to the study of liquids and of gases separately. The study of liquids in equilibrium is hydrostatics (ὑδωρ = water, στασις = standing); that of gases in equilibrium, pneumatics (πνευμα = breath). Intermediate to these will be studied specific gravity.

CHAPTER XVIII.

II. HYDROSTATICS.

In Hydrostatics we will consider first certain especial applications to liquids of the two great principles already studied, and next floating bodies.

A. *Application of principle* (1) *to liquids.*—Of the instruments in which principle (1) is utilized, the best known is

The Hydraulic or Bramah Press.

This press consists of two cylinders, one much larger than the other, connected by a tube. In each works a plunger. The smaller plunger is moved up and down by a lever, and the cylinder wherein it moves is connected below with a reservoir of water. The top of the larger plunger is wide and flat, and upon it the substances to be pressed are placed. If the larger piston be forced upwards, the substances on its summit will be pressed against the top of the immovable framework up and down which the piston-top moves. There are valves at the bottom of the smaller cylinder, and at the end of the connecting pipe next the large cylinder, opening in such a direction as to allow water to flow towards but not from the larger cylinder.

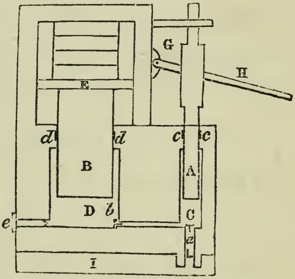


Fig. 110.

The action of the hydraulic press is as follows. The smaller piston, forced down by the lever, drives water into the larger cylinder, and transmits pressure through the water to the larger piston. Just so many times as the area of the small piston is contained in the area of the large, so many times will the pressure applied through the lever to the small be multiplied in its transmission to the large. Suppose the radius of the smaller piston is 4 decimetres, and that of the larger 20 decimetres, and a pressure of one kilogram is exerted on the former. As the area of a circle $= \pi r^2$, the two areas are as the squares of the radii, therefore as 16 to 400.

On an area represented by 16, the pressure is 1 kilogram;
" " " " 1, " " " $\frac{1}{16}$ "
" " " " 400, " " " $\frac{1}{16} \times 400$
$= 25$ kilograms.

In this machine, as in all, the works done by the two pistons are equal. In the case just given, suppose the small piston descends one decimetre.

Let us see how much water it would force into the larger cylinder in doing this.

Its area $= 16 \times \frac{22}{7}$ square decimetres. Descending through 1 decimetre in height, $16 \times \frac{22}{7} \times 1$ cubic decimetres of water are forced into the large cylinder.

The area of the large cylinder $= \frac{22}{7} \times 400$ square decimetres.

The number of cubic decimetres of water forced in divided by this area will give the height to which the piston is raised.

$$\frac{22}{7} \times 16 \times 1 \div \left(\frac{22}{7} \times 400\right) = \frac{\cancel{22}}{\cancel{7}} \times 16 \times 1 \times \frac{\cancel{7}}{\cancel{22}} \times \frac{1}{400} = \frac{1}{25} \text{ decimetres.}$$

The work done by the small piston is therefore represented by

$$\frac{22}{7} \times 16 \times 1 = \frac{352}{7}.$$

The work done by the large piston is therefore represented by

$$\frac{22}{7} \times 400 \times \frac{1}{25} = \frac{352}{7}.$$

Example 53.—In a Bramah press, what must be the circumference of the large piston if a force of 10 grams applied to the smaller piston causes a pressure = 14 kilograms on the large when the radius of the small piston is 7 centimetres?

Given a small piston, with radius 7 centimetres, pressed down by a force of 10 grams, and causing a pressure on the large piston of 14,000 grams.

It is required to find the circumference of the large piston.

The pressure of 10 grams is on an area represented by $7 \times 7 \times \frac{22}{7} = 154$ square centimetres.

The pressure of 1 gram is on an area represented by $\frac{154}{10}$.

The pressure of 14,000 grams is on an area represented by $\frac{154}{10} \times 14000 = 154 \times 1400$.

Area $= 154 \times 1400$. But area $= \pi r^2 = \frac{22r^2}{7}$;

$$\therefore \frac{22r^2}{7} = 154 \times 1400;$$

$$\therefore r^2 = 154 \times 1400 \times \frac{7}{22} = 7^2 \times 1400;$$

$$\therefore r = \sqrt{7^2 \times 1400} = \sqrt{7^2 \times 100 \times 14} = 70\sqrt{14}.$$

$$\begin{aligned} \text{Circumference} &= 2\pi r \\ &= \frac{22}{7} \times 2 \times 70\sqrt{14} \\ &= 44 \times 10\sqrt{14} \\ &= 440\sqrt{14}. \end{aligned}$$

B. *Application of principle* (2) *to liquids.*—Founded upon principle (2) is

(*a*) The Hydrostatic Bellows.

A man stands upon the upper of two boards that are joined by corrugated, water-tight leather. A tube open above communicates below with the cavity of the bellows. Pouring water down this tube, he is able to raise himself; for, applying principle (2), whatever the height of the column of water in the tube, the area of its base is practically equal to the area of the board whereon he stands. As soon, therefore, as the height of the water poured into the tube is such that a column of water of that height and of area equal to the area of CD would have a weight a little greater than that of the man, the latter begins to rise.

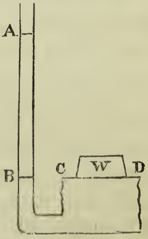


Fig. 111.

Example 54.—In the hydrostatic bellows, to what height must the water have risen in the tube, if a man of weight 70 kilograms is just supported on a circular board, of circumference 12 decimetres?

Given that the board has a circumference of 120 centimetres, and that the man weighs 70,000 grams.

It is required to find x, the height of the water in the tube.

$$\text{Circumference} = 2\pi r,$$

$$120 = \frac{44}{7} r,$$

$$r = 120 \times \frac{7}{44} = \frac{30 \times 7}{11},$$

$$\text{and area} = \left(\frac{30 \times 7}{11}\right)^2 \times \frac{22}{7}.$$

$$\text{Let height} = x, \text{ then } \frac{30 \times 30 \times 7 \times 7}{11 \times 11} \times \frac{22}{7} \times x = 70000 \text{ grams},$$

$$\frac{12600x}{11} = 70000,$$

$$\frac{126x}{11} = 700,$$

$$\frac{18x}{11} = 100,$$

$$x = 100 \times \frac{11}{18}$$

$$= 50 \times \frac{11}{9}$$

$$= \frac{550}{9} = 61\frac{1}{9} \text{ cm.}$$

(*b*) Further, from principle (2) it follows that so long as the depths and the base areas of any number of vessels are equal, the pressure of the same fluid within them on the respective bases will be the same, no matter what the shape of the vessels may be.

In the vessels A, B, C, the depths of the centre of gravity of all the three bases are equal, and the areas of all the three bases are equal. If the three vessels, therefore, are filled with the same fluid, the pressures of the fluid upon the three bases will

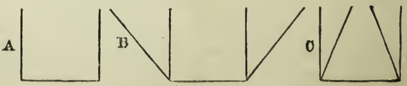


Fig. 112.

be the same. The vertical lines in B and C represent the sides of the columns of fluid exerting pressure upon the bases of B and C; and it will be seen that this column in each case = A. With regard to B, it is easy to conceive the two slanting sides as receiving the pressure due to the two side, triangularly-shaped fluid masses. In C, however, we have the curious case of a pressure upon the base greater than the whole weight of the fluid employed. This increase over the actual fluid weight is due to the reaction of the two slanting sides against the outward pressure of the water. This reaction is transmitted through the water to the base.

(*c*) Lastly, from this second principle it results that if a number of vessels containing the same liquid communicate, the level of the liquid must be the same in all; for the pressure at any point due to all and sundry of the liquid columns must be the same, or equilibrium could not be. If the pressure due to the weight of the liquid in any one vessel were greater than each of the rest of the pressures due to the weights of the liquid in the other vessels, movement must ensue. But these pressures are directly as the depths of the point chosen below the surface of the liquid in each vessel, *i.e.* as the heights of the liquid in the different vessels. As the pressures are equal, the heights of the liquids in the various vessels must be equal also. This is the technical form of the common phrase, 'water seeks its own level.'

C. *Application of principles* (1) *and* (2) *to liquids.*—We may be required to find the total pressure upon an immersed surface when some pressure is being transmitted through the fluid wherein the surface is immersed. This total pressure will be the pressure due to the mere weight of the fluid + the transmitted pressure.

Example 55.—A rectangular vessel, 2 decimetres long, 3 broad, 1 high, is filled with water, and a square piston whose diameter is $\sqrt{8}$ centimetres is let into the roof. If upon this piston is placed a weight of 4 grams, calculate the whole pressure on the base of the box and on one of its longest sides.

Given a rectangular vessel 20 cm. long, 30 broad, and 10 high, filled with water, and a square piston with diameter $\sqrt{8}$ weighted with 4 grams.

It is required to find the whole pressure on the base and on one of the longer sides. We have to consider the weight of the water and the pressure transmitted through the water.

(1) On base.—Weight of the water $= 20 \times 30 \times 10$ grams $=$ 6000 grams.

To find the side of the piston: $(\sqrt{8})^2 = 2x^2$, $8 = 2x^2$, $4 = x^2$ $x = 2$.

On area of 4 sq. cm. is a weight of 4 grams;

$\therefore$ weight on each sq. cm. $= 1$ gram;

$\therefore$ pressure of 1 gram is transmitted to every sq. cm. of surface of base $= 20 \times 30 \times 1 = 600$.

Total pressure $= 6000 + 600 = 6600$ grams.

(2) On side.—The weight of the water $= 30 \times 10 \times 5 = 1500$ grams.

Transmitted pressure on every sq. cm. $= 1$ gram $= 30 \times 10 =$ 300 grams.

Total pressure $= 1500 + 300 = 1800$ grams.

D. *Bodies wholly or partially immersed in liquids.*—When a body is completely immersed in a liquid, it displaces its own volume of that liquid. When a body is partially immersed in a liquid, it displaces a volume of the fluid equal to the volume of so much of the body as is immersed. This displaced liquid tries to regain the position whence it has been displaced. To do this it must press upon the immersed body. The displaced liquid does so press upon the immersed body with all its force, *i.e.* with a force equal to the weight of the displaced liquid. Hence we have two forces acting—(1) the weight of the immersed body acting vertically downwards; (2) the weight of the displaced liquid acting vertically upwards. One of three conditions may obtain.

(*a*) (1) may be greater than (2). The body sinks.

(*b*) (1) may be equal to (2). The body rests in any position in the liquid, provided the vertical line through its centre of gravity passes through the centre of gravity of the displaced liquid. This last proviso is necessary, as two forces to be in equilibrium must not only be equal but opposite, and as the weight of the body may be considered as acting at its centre of gravity, and that of the fluid displaced at *its* centre of gravity, these two centres must be in the same vertical line.

(*c*) (1) may be less than (2). The body will then be forced upwards, but not for ever. For, as it is thus forced upwards, (1), its own weight alters not; but (2), the weight of the water displaced, lessens until at last (1) and (2) are equal, and the body floats. This floating must evidently occur when the weight of the whole body equals the weight of a volume of fluid, equal not to the volume of the whole body, but to the volume of the part immersed. Therefore for bodies floating in a liquid, and only partly immersed, the conditions are—

(i.) That the weight of the whole body equals the weight of the liquid displaced.

(ii.) That the centre of gravity of the liquid displaced and that of the immersed portion of the solid are in the same vertical line. This centre of gravity of the immersed part of the solid is often called the centre of buoyancy.

Case (*a*) would be illustrated by a piece of iron in water. The weight of the iron would be greater than that of the water displaced, and the iron would sink.

Case (*b*) would be illustrated by a piece of water in water. The weight of any mass of water would be equal to the weight of the same volume of the water wherein it is immersed, and the mass of water would have no tendency to change its place.

Case (*c*) would be illustrated by a piece of cork in water. The cork would float when its own weight and that of a volume of water equal to the volume of the cork immersed were equal, and the centre of buoyancy of the cork and the centre of gravity of the displaced water were in the same vertical line.

Formulæ.—Circumference of circle $= 2\pi r$. Area of circle $= \pi r^2$ Area of surface of sphere $= 4\pi r^2$. Contents of sphere $= \frac{4}{3}\pi r^3$.

EXERCISES.

1. Into a regular cubical vessel whose edge is $3\frac{1}{2}$ decimetres, a piston is let of length 3 centimetres, breadth 4 centimetres. What will be the transmitted pressure to a piston occupying one-fifth of one of the sides of the vessel, if $\frac{3}{7}$ gram is placed on the first piston?

2. What will be the edge of a regular cubical vessel if when a circular piston of radius $\sqrt{\frac{63}{2}}$ centimetre is pressed in by a weight = 110 grams, the pressure transmitted to the base is 1 kilogram?

3. A vessel whose sides are, length 20 decimetres, breadth 10 decimetres, height 5, is filled with water. Compare the pressure on the base and on the several sides.

4. A box whose section is an equilateral triangle, 40 decimetres in the side, contains a fluid twice as heavy as water. If the

pressure on the base is $10\sqrt{3}$ kilograms, what is the depth of the fluid in the box?

5. Into the top of a vessel whose length, breadth, and height are respectively 1 metre, $\frac{1}{2}$ a metre, $\frac{1}{4}$ of a metre, a piston of diameter 1·4 decimetres is fixed. If the fluid in the box be water, and the piston be forced in by a weight of 1 gram, calculate the whole of the pressure upon the interior of the box.

6. A sphere of radius 3 decimetres is filled with water. A circular piece of radius 1 centimetre is removed from its surface, and a body $= \frac{1}{4}$ of a gram in weight is placed in the aperture What is the pressure upon the interior of the sphere?

CHAPTER XIX.

III. SPECIFIC GRAVITY.

A. Meaning of the Phrase.

It has been seen that all bodies tend to move towards the earth with a certain force. The attraction towards the earth is known as one of the forms of gravitation. The result of that attraction experienced when the body is held in the hand and not allowed to fall, is called the weight or gravity of the body. Gravity, therefore, means weight.

Hitherto we have only spoken of the absolute weight of bodies, as when we have dealt with pulleys or levers weighing 10 or 5 grams, and so forth. At present we have to do not with absolute weight, but relative weight. We are now not to be concerned with the inquiry as to what things weigh, so much as with the inquiry as to whether they are heavier or lighter than one another, as to what are their relative weights. Thus a grain of iron or a thunderbolt of iron of the same kind will have the same relative weight, though their absolute weights would be very different. Each of them would be about seven times as heavy as distilled water, although the one would weigh a fraction of a gram, and the other possibly many hundreds of kilograms.

Specific, then, in the phrase 'specific gravity,' may be read as relative. Specific gravity will mean to us henceforth relative weight. It is clear that in comparing the relative weights of bodies some standard of reference must be used, and then our definition of specific gravity will run 'the weight of a body in relation to that of the standard.' For gases, the standard is hydrogen. This is of value, as being in the first place a simple gas, and in the second place the lightest gas known. At times air is taken as the reference for gases. But it is a mixture, and not an element; and as it is 14·46 times as heavy as hydrogen, awkward decimals occur. With liquids and solids, distilled water at 4° C., and under a pressure equal to the weight of 760 mm. of mercury, is taken as standard.

B. General Principle.

One general principle will carry the student throughout the whole of the study of specific gravity, and he is strongly advised to keep this before him in every case where specific gravity is concerned.

Specific gravity is relative weight. It is the ratio of the weight of a body to the weight of the standard. To compare these two weights, the same *volume* of the body and of the standard must be taken. It would never do to compare the weights of a large volume of the one and of a small volume of the other. The specific gravity of a body will be the ratio of the weights of a definite volume of it and of the standard. Again, relationship of two quantities or their ratio is always expressed by dividing the one by the other. Thus the ratio of a quantity a to a quantity b is represented by the quotient of a divided by b, or by $\frac{a}{b}$. Hence the ratio of the weights of a certain volume of the body and of the same volume of the standard is expressed by dividing the former weight by the latter. And, finally, *the specific gravity of a body is found by dividing the weight of a given volume of the body by the weight of the same volume of the standard.*

In every case, therefore, find the weight of a known volume of the body whose specific gravity is required. Then find the weight of the same volume of the standard. Divide the former by the latter.

C. Methods of determining Specific Gravity.

We shall consider the determination of the specific gravity of gases, of liquids, of solids.

(1) Of Gases.

(*a*) *Whose chemical composition is known.*

If a gas is a chemical element or a chemical compound whose exact composition is known, its specific gravity can be calculated without experiment.

(i.) *Chemical elements.*—It is known that

1 gram of hydrogen at 0° and under a pressure = 760 mm. mercury measures 11·19 litres.
16 grams of oxygen ,, ,, ,, ,, ,,
14 ,, nitrogen ,, ,, ,, ,, ,,
35½ ,, chlorine ,, ,, ,, ,, ,,

1, 16, 14, 35½ are the respective weight numbers of the gases hydrogen, oxygen, nitrogen, chlorine. The number of grams of each of these represented by its weight number under normal temperature and pressure, measure roughly 11·2 litres. And this holds good of element gases generally. If as many grams of an element gas as are represented by its weight number are taken at 0° and under a pressure equivalent to 760 mm. of mercury, they will occupy 11·2 litres. Hence as these weight numbers give the weight in grams of a certain volume, viz. 11·2 litres, and the weight of the same volume of the standard gas, hydrogen, is 1 gram, the specific gravity of an element gas $= \frac{\text{its weight number}}{1} = \text{its weight number}$.

If air be taken as the standard, and it be remembered that air is 14·46 times as heavy as hydrogen, the specific gravity of an element gas referred to air = its weight number ÷ 14·46.

(ii.) *Chemical compounds.*—It is known that

36½ grams of hydrogen chloride (HCl) at 0° and under pressure = 760 mm. mercury measure 22·38 litres.
17 ,, ammonia (H^3N) ,, ,, ,, ,,
44 ,, carbon dioxide (CO^2) ,, ,, ,,

36½ ($H=1$, $Cl=35\frac{1}{2}$), 17 ($H^3=3$, $N=14$), 44 ($C=12$, $O^2=32$) are the respective weight numbers of the gases hydrogen chloride, ammonia, carbon dioxide. The number of grams of each of these represented by its weight number under normal temperature and pressure will measure roughly 22·4 litres. And this holds good of compound gases generally. If as many grams of a compound gas as are represented by its weight number are taken at 0°, and under a pressure equivalent to 760 mm. of mercury, they will occupy 22·4 litres. Hence as these weight numbers give the weight in grams of a certain volume, viz. 22·4 litres, and the weight of the same volume of the standard gas, hydrogen, is 2 grams,

$$\text{the specific gravity of a compound gas} = \frac{\text{its weight number}}{2}.$$

(*b*) *Whose chemical composition is not known.*

In the case of mixtures, or of unknown elements or compounds, as in the case of the known elements and compounds ere the above generalizations were established, the following plan is adopted.

Weigh very carefully in vacuo a known volume, say 1 litre, of the gas whose specific gravity is required. Then weigh in vacuo the same volume of hydrogen gas, taking care that these weighings are effected with the gases at precisely the same temperature and under precisely the same pressure. The first weight divided by the second gives the specific gravity of the gas. The best temperature and the best pressure to be used are respectively 0° and 760 mm., and if 11·2 litres of the gas can be obtained, that is the best volume to be taken, as 11·2 litres of hydrogen at 0° and under 760 mm. weigh 1 gram, and the divisor in the calculation becoming 1, the actual weight of the 11·2 litres of the gas *is* its specific gravity.

Example 56.—70 grams of an element gas at 0° and under 760 mm. occupy 56 litres. Find the specific gravity of the gas on the hydrogen scale, and on the air scale.

Given that 70 grams of an element gas at 0° and 760 mm. occupy 56 litres. It is required to find the specific gravity of the gas on the hydrogen, and on the air scales.

11·2 litres of hydrogen weigh 1 gram,

56 litres of the gas weigh 70 grams,

1 litre of the gas weighs $\frac{70}{56} = \frac{10}{8}$,

11·2 litres of the gas weigh $\frac{10}{8} \times 11{\cdot}2 = \frac{112}{8} = 14$.

The relative weights, therefore, of equal volumes of the gas and of hydrogen are as 14 and 1.

$$\text{Ratio} = \frac{14}{1}.$$

Specific gravity of gas on the hydrogen scale = 14.

On the air scale the weight of 11·2 litres of the gas must be divided not by 1, but by 14·46, air being 14·46 times as heavy as hydrogen.

$$\text{Ratio} = \frac{14}{14{\cdot}46} = \frac{7}{7{\cdot}23}.$$

Example 57.—Marsh gas has formula CH^4($C = 12, H = 1$). Find its specific gravity on the hydrogen scale and on the air scale.

Given that the weight number of marsh gas, CH_4, is 16, it is required to find its specific gravity on the hydrogen, and on the air scales.

22·4 litres of this compound gas weigh 16 grams,

1 litre of this compound gas weighs $\frac{16}{22{\cdot}4}$,

11·2 litres of this compound gas weigh $\frac{16 \times 11{\cdot}2}{22{\cdot}4} = \frac{16}{2} = 8$.

The relative weights ∴ of equal volumes of marsh gas and of hydrogen are as 8 : 1.

$$\text{Ratio} = \frac{8}{1}, \text{ specific gravity of marsh gas} = 8.$$

The weight of 11·2 litres of the gas must be divided, as before, by 14·46 to find its specific gravity on the air scale.

$$\text{Ratio} = \frac{8}{14{\cdot}46} = \frac{4}{7{\cdot}23}.$$

Example 58.—20 litres of a gas measured at 0° and 760 mm. weigh $25\frac{23}{28}$ grams. Find its specific gravity on the two scales.

Given that 20 litres of the gas weigh $25\frac{23}{28}$ grams.

It is required to find its specific gravity on the hydrogen, and on the air scales.

20 litres of the gas weigh $25\frac{23}{28}$, or $\frac{723}{28}$ grams.

1 litre of the gas weighs . . . $\frac{723}{28} \div 20$ grams.

11·2 litres of the gas weigh $\dfrac{723 \times 11{\cdot}2}{28 \times 20} = \dfrac{723 \times 2{\cdot}8}{28 \times 5} = \dfrac{723}{50} = 14{\cdot}46.$

The relative weights ∴ of equal volumes of the gas and of hydrogen are as 14·46 : 1.

$$\text{Ratio} = \frac{14{\cdot}46}{1} = 14{\cdot}46.$$

On the air scale the weight of 11·2 litres of the gas must be divided as before by 14·46.

$$\text{Ratio} = \frac{14{\cdot}46}{14{\cdot}46} = 1.$$

(2) Of Liquids.

(*a*) *By the specific gravity bottle.*—A special flask whose weight when empty is known, is filled up to a definite mark with the liquid. The flask and liquid are then weighed. Subtracting from this weight of the two together the weight of the flask, the remainder is the weight of a definite volume of the liquid. Fill the flask next with distilled water at 4° and under a pressure = 760 mm. The flask and the water are then weighed. Subtracting from this weight of these two together the weight of the flask, the remainder is the weight of a definite volume of water. Divide the first remainder by the second, and the quotient is the specific gravity of the liquid.

In using constantly the same specific gravity flask, the weight of the contained distilled water when once ascertained is ascertained for all experiments.

In symbols.—If b = the weight of the bottle,
w = ,, ,, and the liquid,
w' = ,, ,, ,, water,

$$\text{Specific gravity of liquid} = \frac{w-b}{w'-b}.$$

Example 59.—If the specific gravity bottle weighs 5 centigrams alone, when full of water 1 gram, and when full of another liquid 195 centigrams, what is the specific gravity of the liquid?

Given a specific gravity bottle weighing 5 centigrams when empty, 100 centigrams when full of water, and 195 centigrams when full of another liquid, it is required to find the specific gravity of the liquid.

$$\begin{aligned}\text{Specific gravity of liquid} &= \frac{w-b}{w'-b}\\ &= \frac{195-5}{100-5}\\ &= \frac{190}{95} = 2.\end{aligned}$$

(*b*) *By weighing a heavy solid in the liquid and in water.*—The student's closest attention is required here, as the principles involved are the same as those involved in one of the methods of obtaining the specific gravity of solid bodies. He is once more reminded that the great principle is to find the weight of a known volume of the liquid, and divide that by the weight of an equal volume of water.

Weigh a solid in vacuo. Then weigh it in the liquid. It will weigh less than it did in vacuo; for the 'sinker' displaces its own volume of the liquid. This displaced liquid strives to regain its position with a force = its weight. Hence the sinker is pressed up by a force = the weight of its own volume of the liquid. Its apparent loss of weight then = the weight of a volume of the liquid equal to the volume of the sinker. Note this loss of weight in the liquid.

Next weigh the sinker in distilled water. It will weigh less than it did in vacuo; for the sinker displaces its own volume of the water. This displaced water strives to regain its position with a force = its weight. Hence the sinker is pressed up by a force = the weight of its own volume of the water. Its apparent loss of weight = the weight of a volume of water equal to the volume of the sinker. Note this loss of weight in the water.

Lastly, as the sinker's loss of weight in the liquid is the weight of a volume of the liquid equal to the volume of the sinker, and as its loss of weight in the water is the weight of a volume of water equal to the volume of the sinker, we have now obtained the weight of two equal volumes of the liquid and the sinker, and the specific gravity of the former will be found by dividing the loss of weight of the sinker in the liquid by the loss of weight of the sinker in water.

In symbols.—If the solid weigh s in vacuo,
,, ,, l in the liquid,
,, ,, w in the water,

$$\text{Specific gravity of liquid} = \frac{s-l}{s-w}.$$

Example 60.—A solid weighs in vacuo 10 grams, in water 8 grams, in another liquid 6 grams. Find the specific gravity of the liquid.

Given that the solid weighs 10 grams in vacuo, 8 grams in water, and 6 grams in another liquid.

It is required to find the specific gravity of the liquid.

$$\text{Specific gravity of the liquid} = \frac{s-l}{s-w}$$

$$= \frac{10-6}{10-8} = \frac{4}{2} = 2.$$

Example 61.—A sinker weighs 12 grams in vacuo, and in water 8 grams. What will be its weight in a fluid of specific gravity $\frac{1}{2}$?

Given that a sinker weighs 12 grams in vacuo and 8 grams in water. It is required to find its weight in a fluid of specific gravity $\frac{1}{2}$.

The upward pressure of such a fluid is only half that of water; any quantity of it will ∴ only support half as much weight as the same quantity of water.

The water presses the sinker upwards with a force = 4 grams;
∴ the liquid ,, ,, ,, = 2 grams;
∴ the weight of the body in the liquid will be 12 − 2 = 10 grams.

Or as specific gravity of the liquid $= \dfrac{s-l}{s-w}$,

$$\tfrac{1}{2} = \frac{12-l}{12-8},$$

$$2 = 12 - l,$$

$$l = 10.$$

(*c*) *By hydrometers.*—Hydrometers are light instruments that can be made to float in an upright position in various liquids. There are two kinds—(i.) Those that are of constant weight and whose amount of immersion varies. (ii.) Those that are always made to sink down into the liquid to the same level, that is, whose amount of immersion is constant, and whose weight therefore varies.

(i.) The common hydrometer is always of the same weight. It will sink therefore to different levels in different liquids. To determine by its aid the specific gravity of a liquid, place the hydrometer in the liquid and note the volume immersed. Then place it in distilled water and note the volume immersed. In each case the weight supported by the upward pressure of the displaced liquid or of the displaced water is the same, viz. the weight of the whole hydrometer. Consequently the *weights* of the two different volumes of the liquid and of water displaced are the same. What we require, however, is *the ratio between the different weights of equal volumes.* This ratio will be the inverse form of the ratio between the two different volumes of the same weight. If, for example, the hydrometer sank in a liquid to a certain level and in water half as far, the ratio between the volumes of the liquid and of water whose weights were equal to each other (and to the weight of the hydrometer) would be 2 : 1. But the ratio between the weights of equal volumes of the liquid and of water would be 1 : 2. Hence, to find the specific gravity of a liquid, immerse the hydrometer therein and note the volume of the instrument immersed. Immerse the hydrometer in water and note the volume of the instrument immersed. Divide the latter volume by the former.

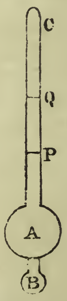


Fig. 113.

In symbols.—If l = volume of hydrometer immersed in liquid,
If w = volume of hydrometer immersed in water,

$$\text{Specific gravity of liquid} = \frac{w}{l}.$$

(ii.) This form of hydrometer, commonly known as Nicholson's hydrometer, is always made to sink to the same level in every liquid. It is therefore of different weights at different times. B is a flat tray always out of the liquid, whereon weights can be placed. A is a light bulb, and C a heavy bulb or a heavy tray and cage. In using the instrument, A and C are always in the liquid. To find the specific gravity of a liquid by this constant immersion hydrometer, place the instrument in the liquid, and place weights on the tray B until the hydrometer is sunk down to the level of the fixed mark D. The weight of the hydrometer and the added weights = the weight of the volume of liquid displaced. Note this weight.

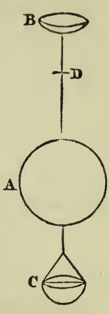


Fig. 114.

Now place the instrument in water and place weights on the tray B until the hydrometer is sunk down to the level of the fixed mark D. The weight of the hydrometer and the added weights in this case = the weight of the volume of water displaced. Note this weight. The volumes of the liquid and of the water displaced are the same in the two cases, as the hydrometer is immersed in both to exactly the same extent. Hence, to find the specific gravity of the liquid, divide the sum of the weight of the hydrometer and the added weights in the first case by the sum of the weight of the hydrometer and the added weights in the second case.

In symbols.—If the weight of the hydrometer $= h$,
If the weights added when immersed in liquid $= l$,
” ” ” water $= w$,

$$\text{Specific gravity of liquid} = \frac{h+l}{h+w}.$$

Example 62.—With a variable immersion hydrometer it is found that the ratios of the volumes immersed in water, in a liquid (α) and in a liquid (β), are respectively $\frac{1}{2}$, $1\frac{1}{2}$, $\frac{1}{8}$. Find the specific gravity of (α) and of (β), and state what would be the specific gravity of (α) if that of (β) were taken as unity.

Given a hydrometer, the ratio of the volumes immersed being $\frac{1}{2}$ in water; in liquid (α), $\frac{3}{2}$; in liquid (β), $\frac{1}{8}$.

It is required to find the specific gravity of (α) and (β), and to state the specific gravity of (α) if (β) be taken as 1.

The amount of immersion of the hydrometer varies inversely as the specific gravity of the liquid.

In liquid (α) the immersion is as $3 : 1$, taking immersion in water as 1; $\therefore$ specific gravity is as $1 : 3 = \frac{1}{3}$.

In liquid (β) the immersion is as $\frac{1}{4} : 1$; $\therefore$ specific gravity is as $4 : 1 = \frac{4}{1} = 4$.

$$\text{Or, specific gravity of } (\alpha) = \frac{w}{l} = \frac{\frac{1}{2}}{\frac{3}{2}} = \frac{1}{3},$$

$$\text{,,} \qquad \text{,,} \qquad (\beta) = \frac{w}{l} = \frac{\frac{1}{2}}{\frac{1}{8}} = \frac{8}{2} = 4.$$

If (β) be taken as unity, then this last specific gravity 4 is now represented by 1.

$$\therefore 1 \text{ is now represented by } \tfrac{1}{4}.$$

$$\therefore \tfrac{1}{3} \qquad \text{,,} \qquad \text{,,} \qquad \tfrac{1}{4} \times \tfrac{1}{3} = \frac{1}{12}.$$

Example 63.—A Nicholson's hydrometer weighing 10 grams is weighted with 20 grams and with 30 grams respectively when immersed in water and another liquid. Find the specific gravity of the liquid. With what must the instrument be weighted to sink down to the usual level in absolute alcohol (specific gravity ·8)?

Given a Nicholson's hydrometer weighing 10 grams, and weighted with 20 grams and 30 grams respectively when immersed in water and in a given liquid.

It is required to find the specific gravity of the liquid, and also to find the weights required to sink the hydrometer to the usual level in absolute alcohol whose specific gravity = ·8.

$$\text{Specific gravity of the liquid} = \frac{h+l}{h+w}$$

$$= \frac{10+30}{10+20} = \frac{40}{30} = \frac{4}{3} = 1\tfrac{1}{3} \text{ or } 1{\cdot}\dot{3}.$$

$$\text{Specific gravity of alcohol } {\cdot}8 = \frac{10+l}{10+20},$$

$$24 = 10 + l,$$

$$24 - 10 = l,$$

$$14 = l;$$

$\therefore$ weights = 14 grams would be required.

(3) Of Solids.

(*a*) *Heavier than water, and insoluble therein.*—Weigh the solid in vacuo. Weigh it in water. From that which has already been shown, the solid will weigh less by the weight of a volume of water equal to its own volume. We have therefore the two requisites, viz. the weight of a certain volume of the solid in vacuo, and the weight of an equal volume of water. This last weight will be the loss of weight of the solid in water. To find the specific gravity of the solid, divide its weight in vacuo by its loss of weight in water.

In symbols.—If s = weight of solid in vacuo,
s' = " " water,

$$\text{Specific gravity of solid} = \frac{s}{s - s'}.$$

Example 64.—A solid weighs in vacuo 25 grams, in water 20 grams. Find its specific gravity.

Given a solid which weighs 25 grams in vacuo, and 20 grams in water.

It is required to find its specific gravity.

The specific gravity of a substance equals its own weight divided by the weight of an equal bulk of the standard, *i.e.* in this case, by its loss of weight in the standard.

The weight of the solid is 25 grams; the loss of weight in water is 5 grams;

$$\therefore \text{ specific gravity of solid} = \frac{25}{5} = 5.$$

$$\text{Or in symbols, } \frac{s}{s - s'} = \frac{25}{25 - 20} = \frac{25}{5} = 5.$$

Example 65.—A solid of specific gravity 7 weighs in vacuo 42 grams. Find its weight in water.

Given a solid whose specific gravity is 7, and whose weight in vacuo is 42 grams.

It is required to find its weight in water.

$$\text{Using the same symbols, specific gravity} = \frac{s}{s - s'},$$

$$7 = \frac{42}{42 - s'},$$

$$1 = \frac{6}{42 - s'},$$

$$42 - s' = 6,$$

$$s' = 42 - 6 = 36.$$

(*b*) *Heavier than water, and soluble therein.*—Weigh the solid in vacuo. Then weigh it in a liquid of known specific gravity.

Find its loss of weight in this liquid, and knowing the specific gravity of the liquid, calculate from the loss of weight therein what would be the loss of weight in water.

Example 66.—A solid weighing in vacuo 18 grams weighs in alcohol (specific gravity, ·8) $12\frac{3}{5}$ grams. What is its specific gravity?

Given a solid which weighs 18 grams in vacuo, and $12\frac{3}{5}$ in alcohol of specific gravity ·8.

It is required to find its specific gravity.

Loss of weight in alcohol $= 18 - 12\frac{3}{5} = 5\frac{2}{5}$ grams;

,, ,, water, therefore, $= \frac{27}{\cancel{5}} \times \frac{\overset{2}{\cancel{10}}}{\underset{4}{\cancel{8}}} = \frac{27}{4} = 6\frac{3}{4}$ grams.

$$\text{Specific gravity} = \frac{\text{weight in vacuo}}{\text{loss of weight in water}}$$

$$= \overset{2}{\cancel{18}} \times \frac{4}{\underset{3}{\cancel{27}}} = \frac{8}{3}$$

$$= 2\frac{2}{3} = 2\cdot\dot{6}.$$

Example 67.—Find the weight of a mass of a solid of specific gravity 4 when it is weighed in a fluid whose specific gravity is $\frac{1}{2}$, if in vacuo the mass weigh 40 grams.

Given a solid of specific gravity 4, whose weight in vacuo is 40 grams.

It is required to find its weight in a fluid of specific gravity $\frac{1}{2}$.

$$\text{Specific gravity} = \frac{\text{weight in vacuo}}{\text{loss of weight in water}}.$$

Let loss of weight in water $= x$;

then $4 = \frac{40}{x}$, $4x = 40$, $x = 10$.

Loss of weight in water $= 10$;

$\therefore$ upward pressure of water $= 10$;

$\therefore$ upward pressure of fluid of specific gravity $\frac{1}{2} = 5$.

Weight in fluid $= 40 - 5 = 35$ grams.

(*c*) *Lighter than water, and insoluble.*—The difficulty in this case is that it is impossible by the preceding method to find the weight of a volume of water equal in volume to the bulk of the light body. The latter floats, and therefore appears to weigh nothing when weighed in water. The following plan is adopted :—

Weigh the light solid in vacuo. Take a piece of some heavy substance, as lead. Weigh this first in vacuo, second in water. The difference equals the weight of a volume of water equal to the volume of the lead. Next weigh the light and the heavy body together in vacuo and in water. The difference in this case equals the weight of a volume of water equal to the volume of the two bodies taken together. The difference between these differences is the weight of a volume of water equal to the volume of the light body. Divide the weight of the latter in vacuo by this last quantity, and we have the specific gravity of the light body.

In symbols.—If $w =$ weight of light substance in vacuo,
$h =$,, heavy ,, ,,
$h' =$,, ,, ,, in water.
$b =$,, both substances ,,
Then $w + h =$ weight of both in vacuo,
$w + h - b =$,, equal volume of water to both,
$h - h' =$,, ,, ,, ,, heavy body,
$(w + h - b) - (h - h') =$ weight of equal volume of water to light body.

$$\text{Specific gravity of light substance} = \frac{w}{(w + h - b) - (h - h')}.$$

Example 68.—A piece of cork weighing 24 grams is attached to a piece of metal that weighs in vacuo 800 grams, and in water 720 grams, and the two together weigh in water 600 grams. Find the specific gravity of the cork.

Given a piece of cork weighing 24 grams, attached to a piece of metal weighing 800 grams in vacuo and 720 in water, the two together weighing 600 grams in water.

It is required to find the specific gravity of the cork.

Weight of water equal in bulk to the metal $= 800 - 720 = 80$.
,, ,, ,, ,, and the cork $= 824 - 600 = 224$;

$\therefore$ weight of water equal in bulk to the cork only $= 224 - 80 = 144$.

Specific gravity $= \dfrac{\text{weight in vacuo}}{\text{weight of equal bulk of standard}} = \dfrac{24}{144} = \dfrac{2}{12}$ $= \dfrac{1}{6} = \cdot 1\dot{6}$. Or by formula, specific gravity of cork

$$= \frac{w}{(w+h-b)-(h-h')}$$
$$= \frac{24}{(24+800-600)-(800-720)}$$
$$= \frac{24}{224-80}$$
$$= \frac{24}{144}$$
$$= \tfrac{1}{6}.$$

Example 69.—What will be the weight of a piece of wood of specific gravity $\frac{1}{4}$, if, when attached to a piece of metal that weighs 50 grams in vacuo and 40 grams in water, the two together weigh 16 grams in water?

Given a piece of wood whose specific gravity is $\frac{1}{4}$, and a piece of metal whose weight in vacuo is 50 grams and in water 40 grams, the weight of the two in water being 16 grams.

It is required to find the weight of the wood.

Let $x =$ weight of wood.

Loss of weight of metal only in water $= 50 - 40 = 10$.

Loss of weight of both in water $= (50 + x) - 16 = 34 + x$.

Loss of weight of wood only $= 34 + x - 10 = 24 + x$.

$$\text{Specific gravity of wood, } \tfrac{1}{4} = \frac{x}{24+x},$$
$$24 + x = 4x,$$
$$24 = 3x,$$
$$x = 8.$$

$$\text{Or by formula, } \frac{x}{(x+50-16)-(50-40)}$$
$$= \frac{x}{x+34-10}; \therefore \tfrac{1}{4} = \frac{x}{x+24},$$
$$x + 24 = 4x,$$
$$24 = 3x,$$
$$x = 8.$$

(*d*) *Lighter than water and soluble therein.*—The method is similar to the last, but the weighing of the heavy body and of the two together must be conducted in a liquid that will dissolve neither, and whose specific gravity is known. The losses of weight in this known liquid, *i.e.* the weights of volumes of this liquid equal to those of the two substances taken together and to that of the heavier, must be turned into their equivalents in water, and the difference of those equivalents taken to obtain the weight of a volume of water equal to the volume of the light body.

Example 70.—A body lighter than and soluble in water weighs 4 grams in vacuo. Attached to a mass of metal that weighs 24 grams in vacuo and 23 in a fluid of specific gravity $\frac{1}{2}$, the two are found to weigh in this fluid 11 grams. Find the specific gravity of the light body.

Given a body lighter than and soluble in water whose weight in vacuo is 4 grams, and which is attached to a piece of metal weighing 24 grams in vacuo and 23 in a fluid of specific gravity $\frac{1}{2}$, the two weighing in this fluid 11 grams.

It is required to find the specific gravity of the light body.

Loss of weight of metal in fluid of sp. gr. $\frac{1}{2} = 1$ gram;

$\therefore$ in water $= 2$ grams.

Loss of weight of metal and lighter body in fluid of sp. gr. $\frac{1}{2} = 28 - 11 = 17$;

$\therefore$ in water $= 34$ grams.

Loss of weight of lighter body only in fluid of sp. gr. $\frac{1}{2} = 17 - 1 = 16$;

$\therefore$ in water $= 32$ grams.

$$\text{Sp. gr.} = \frac{\text{weight in vacuo}}{\text{loss of weight in water}} = \frac{4}{32} = \frac{1}{8} = \cdot 125.$$

Or by formula,

$$\frac{w}{(w+h-b)-(h-h')} = \frac{4}{(4+24-11)-(24-23)}$$
$$= \frac{4}{28-11-1}$$
$$= \frac{4}{16} \text{ in fluid of sp. gr. } \tfrac{1}{2}$$
$$= \frac{4}{32} \text{ in water}$$
$$= \frac{1}{8}.$$

Example 71.—8 grams of a substance of specific gravity $\frac{1}{7}$ are fastened to 30 grams of a body of specific gravity 15. Find what the two will weigh in a fluid of specific gravity $\frac{1}{2}$.

Given that 8 grams of a substance of specific gravity $\frac{1}{7}$ are fastened to 30 grams of a body of specific gravity 15.

It is required to find what the two will weigh in a fluid of specific gravity $\frac{1}{2}$.

Let $x=$ loss of weight of lighter body in water.

$$\text{Then } \frac{1}{7}=\frac{8}{x},\ x=56\ ;$$

and taking x as loss of weight of heavier body in water,

$$15=\frac{30}{x},\ x=2.$$

Loss of weight of both in water $=58$;

$\therefore$ loss of weight of both in liquid of sp. gr. $\frac{1}{2}=58\times\frac{1}{2}=29$.

Weight in fluid $=38-29=9$ grams.

(*e*) *Powders.*—The difficulty in finding the specific gravity of a powder is, that when the attempt is made to find the weight of the powder in water, the subject of the experiment is liable to diffuse into the water around.

With powders, a bottle containing a known weight of water is taken. After weighing the powder in vacuo, the powder is placed in the bottle previously filled up to the mark with distilled water. Now the powder displaces *its own volume of water.* Remove, by means of blotting-paper, the water, until that which is left stands at the marked level. Then weigh. Subtracting the weight of bottle, powder, and residue water from the weight of bottle, powder, and original water, the remainder is the weight of water displaced, *i.e.* the weight of a volume of water equal to that of the powder. Divide the weight of the powder in vacuo by this remainder, and the quotient is the specific gravity of the powder.

In symbols.—If $p=$ weight of powder in vacuo,
$b=$ „ empty bottle in vacuo,
$w=$ „ water that fills the bottle up to the mark,
$a=$ „ bottle, powder, and residue water,

$$\text{Specific gravity of powder}=\frac{p}{(b+p+w)-a}.$$

Example 72.—A powder weighs in vacuo $\frac{1}{2}$ hectogram. Placed in a bottle that weighs half a gram, and contains, when filled up to the mark, a litre of distilled water, the weight of the whole, when the overflow of water has been removed, is found to be $1040\frac{1}{2}$ grams. What is the specific gravity of the powder?

Given a powder which weighs in vacuo 50 grams, and which is placed in a bottle weighing half a gram, and containing a weight of water = 1000 grams. When the overflow of water is removed, the weight of the whole = $1040\frac{1}{2}$ grams. It is required to find the specific gravity of the powder.

The bottle, water, and powder weigh respectively 50, 1000, $\frac{1}{2}$; in all, $1050\frac{1}{2}$ grams. The bottle, powder, and residue water weigh together $1040\frac{1}{2}$ grams. Weight of water displaced therefore = 10 grams.

But water displaced = in bulk the powder,

$$\text{Specific gravity} = \frac{\text{weight in vacuo}}{\text{weight of equal bulk of standard}} = \frac{50}{10} = 5.$$

Or by formula,

$$\text{Sp. gr.} = \frac{p}{(b+p+w)-a} = \frac{50}{(\frac{1}{2}+50+1000)-1040\frac{1}{2}} = \frac{50}{10} = 5.$$

Exercise 73.—What weight of a powder of specific gravity 4 must be placed in the bottle of the preceding example, that the weight, after removal of the superfluous water, may be $1075\frac{1}{2}$ grams?

Given the same bottle, and a powder of specific gravity 4, it is required to find what weight of powder will make the whole weigh $1075\frac{1}{2}$ grams, after removal of superfluous water.

Let the weight = x. The bottle, water, and powder weigh respectively $\frac{1}{2}$, 1000, x; in all, $1000\frac{1}{2}+x$ grams. The bottle, remaining water, and powder weigh together $1075\frac{1}{2}$. Weight of water displaced therefore = $x-75$.

$$\text{Specific gravity } 4 = \frac{x}{x-75},$$

$$4x-300 = x,$$

$$3x = 300,$$

$$x = 100.$$

Or by formula,

$$\text{Specific gravity} = \frac{p}{(b+p+w)-a} = \frac{x}{(\frac{1}{2}+x+1000)-1075\frac{1}{2}},$$

$$4 = \frac{x}{x-75},$$

$$4x-300 = x,$$

$$3x = 300,$$

$$x = 100.$$

EXERCISES.

1. What is the specific gravity of an element gas 1 litre of which at 0°, and under a pressure of 760 mm. mercury, weighs $1\frac{3}{7}$ grams?

2. If the specific gravity of a compound gas is 22, find the weight of 1 litre of it.

3. What would be the actual weight of a solid of specific gravity 10, if it appeared to weigh in water 90 grams? What would it weigh in a fluid of specific gravity 10?

4. If 4 c.c. of a solid three times as heavy as water are put in a certain liquid, they appear to weigh 10 grams. Find the specific gravity of the liquid.

5. What weight of a metal of specific gravity 8 must be added to 20 c.c. of a body whose specific gravity is $\frac{1}{4}$ to make it float in water in any position?

6. 6 grams of a body of specific gravity $\frac{1}{8}$ are fastened to 120 grams of a body of specific gravity 4. What do they appear to weigh in water?

7. What weight of a powder of specific gravity 3 must be introduced into a litre bottle full of distilled water, that the weight of the whole may be 1020 grams? Weight of bottle may be disregarded.

8. If a cubic decimetre of a solid weighs 3000 grams in water, what will it weigh in alcohol?

CHAPTER XX.

IV. PNEUMATICS.

The study of gases will involve the study of—A. The air and its pressure; B. The barometers; C. The pumps; D. The siphon; E. Boyle's law; F. Air-pumps; G. Condensing syringe.

A. The Air and its Pressure.

The gas most familiar to us is the air. The air, therefore, will be our chief object of study, but the principles established in respect to air will hold for gases generally.

The air, like all fluids, will transmit pressure equally and in all directions. Like all fluids, also, it will have weight. The pressure on any surface immersed in the air will be equal to the weight of a column of air whose transverse section is equal to the area of the immersed surface, and whose height is equal to the depth of the centre of gravity of the surface below the surface of the air. Such pressure is exerted on everything upon the earth. We live at the bottom of a great ocean of air, extending with ever-diminishing density to a height of about 45 miles above the earth. Now the weight of this column of air is such that it exerts an average pressure = 1033 grams weight on every square centimetre, or $14\frac{2}{3}$ lbs. weight on every square inch.

This huge pressure upon our bodies is unnoticed by us, because within us, in our body-cavities, our blood-vessels, the spaces in our connective tissues, is air under the like conditions to those of that pressing upon our external surface. If this internal air were removed, our bodies would collapse, crushed under the weight of the superincumbent atmosphere.

Proofs of this atmospheric pressure are afforded by some of the simpler experiments with air-pumps:—

(1) *The Magdeburg Hemispheres.*—These are two hollow

hemispheres of metal with closely-fitting edges. If by connection of the space between the two with an air-pump, through the medium of a tube and stopcock, the air between them is removed, the separation of the two hemispheres is found to be a matter of extreme difficulty, as the pressure now tending to keep them together, a pressure due to the atmosphere, is more than a kilogram on every square centimetre.

(2) *The withered apple* placed under the receiver of an air-pump, as the air is exhausted, becomes smooth, because, the pressure of the air on its surface being removed, the air within the apple, by its elasticity, causes the depressed skin to rise into its old place.

This pressure of the air is utilized by man in various ways. Perhaps the most important application of it is in the sustaining of columns of liquid whose heights will vary with the varying pressure of the air. This leads us to the consideration of our second subject.

B. Barometers.

(1) *Construction and Action.*—If a wide, clean glass tube, about 800 mm. long, be filled with mercury, and the open end being temporarily closed by the finger, the tube inverted, placed with the open end in mercury, and the finger removed, the column of mercury in the tube will descend, until at last it ceases to do so, and remains stationary. It will then be noticed that its height above the level of the mercury in the outside vessel is about 760 mm. Such a tube under such conditions is the simplest form of the barometer.

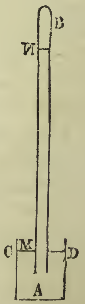


Fig. 115.

The mercury in this tube is sustained by the pressure of the air. If the pressure of the air increase, the column of mercury will rise. If the pressure of the air, from any cause, diminish at the particular part of the earth where our barometer is situated, the column of mercury in the tube will fall to a greater or less extent. Hence the height of this mercury column is a measure of the pressure or weight of the air at any given time and place; hence the name of the instrument (βαρυς = heavy, μετρον = measure).

(2) *Uses.*—Two chief uses are made of the mercury barometer:—

(*a*) To determine heights above the sea-level. As the air is an elastic fluid, its lower layers will be denser than its upper layers, for the lower layers will have a greater mass of air lying upon them than the upper layers; therefore, as we ascend through the air, we find it growing of less and less density, or becoming more and more rare. If we ascend a mountain or go up in a balloon, we notice this increased rarity of the air. We notice it, after a while, in an unpleasant way, for the air within us being still at the same density as the air near the surface of the earth whence we started, presses outwards, as in the shrivelled apple of p. 146, and causes great inconvenience, and even pain.

If, as we ascend through this rarefying air, we carry with us a mercury barometer, the column of mercury therein is found to slowly descend, as at higher levels the supporting pressure of the air is ever less and less. Experiment has determined that with a rise above the sea-level of 900 feet, the mercury column falls a distance of one inch. Thus the barometer may be used to determine the height of a mountain.

(*b*) As a weather-glass. The specific gravity of air on the hydrogen scale $= 14 \cdot 46$. The specific gravity of steam or watery vapour (H^2O) on the hydrogen scale $= 18 \div 2 = 9$. Watery vapour is therefore considerably lighter than air. If, then, any notable quantity of such vapour becomes mixed with the air, the mixture is lighter than the dry air, and, exerting less pressure upon the mercury column in the barometer, that column falls a certain number of millimetres. If the air becomes dry again, it is relatively heavier than during the time it was laden with moisture, exerts more pressure, and the column of mercury in the barometer rises. The presence, therefore, of aqueous vapour in the air is attended with a 'fall of the barometer,' its absence with a 'rise.' When aqueous vapour is present, it is likely to fall as rain; hence a 'fall of the barometer' often portends stormy weather, a rise is often the forerunner of fine weather.

(3) *Forms of Barometer*:—

(*a*) The weather-glass.—In the form of barometer used for the purpose just described, the tube containing the mercury is bent at its lower end, as shown in Fig. 116. In the mercury contained in the short part of the tube next the open end floats a metal weight. From this a string passes over a fixed pulley, carrying an index that moves over the ordinary round face of the instrument as it hangs in the entrance-hall. An increase of pressure of the air causes a rise of the long column of mercury in the closed limb of the instrument, a corresponding fall in the short column of mercury in the open limb, a descent of the metal float, a revolution of the fixed pulley, a passage of the index arrow over the face-plate in the direction in the figure of the hands of a watch. On the other hand, a diminution in the pressure of the air causes a fall of the long column of mercury in the closed limb of the instrument, a corresponding rise in the short column of mercury in the open limb, an ascent of the metal float, a loosening of the cord passing from it over the pulley, a descent of the counterpoise tightening the loosened cord, this causing the pulley to revolve and the index to pass over the face-plate in a direction opposed to that of the hands of a watch.

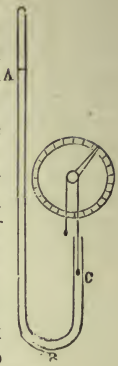


Fig. 116.

(*b*) The Aneroid Barometer.—This form of barometer (α, νηρος = wet, ειδος = resemblance) is made by exhausting a strong metal box that is closed above by a flexible lid of metal. To the lower side of the lid is attached a spring that runs vertically downwards to the floor of the box, and resists the pressure of the air. When that pressure increases, the lid is forced in against the resistant action of the spring. When the pressure of the air diminishes, the lid is forced up by the spring. These movements of the lid are magnified by aid of a rack and pinion (or toothed wheel). To the axis of the latter a long index is attached that moves over a graduated scale.

(*c*) Water Barometer.—Any liquid other than mercury may be employed in the construction of a barometer. The lighter the liquid used, the higher will be the column of the liquid kept up by the pressure of the air. The heavier the liquid used, the shorter will be the column of liquid supported by the air-pressure. In fact, the height of the column, omitting any variations due to alteration in the air density, will vary inversely as the specific gravity of the liquid employed.

The only other liquid that has been used to any extent in making barometers is water. Water-barometers have been made, and are still in use in one or two places. The inconvenience in connection with them is the great height of tube required; for the specific gravity of mercury is about 13·5, that of water being 1. Now, as the heights of columns of liquids supported by the air-pressure vary inversely as the specific gravities of the liquids, and as the air can keep up about 760 millimetres of mercury, it is clear that of water a much greater height will be sustained.

If the air keep up of a liquid of specific gravity $13\frac{1}{2}$ a column whose height is 760 millimetres, the air keeps up of a liquid of specific gravity 1 a column whose height is $760 \times 13\frac{1}{2}$ mm., or $\frac{760}{1} \times \frac{27}{2}$ $= 380 \times 27 = 10{,}260$ mm., or 10 kilometres 2 hectom. 6 decam.

Example 74.—What must be the change of level in a water-barometer when the mercurial one falls from 770 mm. to 743 mm. (sp. gr. of mercury $= 13{\cdot}5$)?

Given that a mercurial barometer falls 27 mm., and that the specific gravity of mercury is 13·5. It is required to find the change of level in a water barometer.

The fall of 1 mm. mercury is equivalent to the fall of 13·5 mm. water;

∴ the fall of 27 mm. mercury is equivalent to the fall of

$$13{\cdot}5 \times 27 = 364{\cdot}5 \text{ mm. water} = 36{\cdot}45 \text{ cm. water.}$$

Example 75.—What must be the specific gravity of a liquid that, when the mercury barometer stands at 750 mm., a barometer made with this liquid may stand at 2025 mm.?

Given that a barometer made with a certain liquid stands at 2025 mm. when the mercurial barometer stands at 750 mm. It is required to find the specific gravity of the liquid.

Let $x =$ the specific gravity of the liquid.

$$\begin{aligned} \text{Then } 750 \times 13{\cdot}5 &= 2025x, \\ 10125 &= 2025x, \\ 405 &= 81x, \\ 45 &= 9x, \\ x &= 5. \end{aligned}$$

C. Pumps for Liquids.

The pressure of the air has much to do also with the useful arrangements known as pumps. The office of these instruments is, as a rule, to raise water to a certain height. Three chief kinds will be studied.

(1) *The Ordinary Pump.*—This is sometimes known as the lifting or suction pump.

(*a*) Construction.—A diagram of a vertical section of such a pump

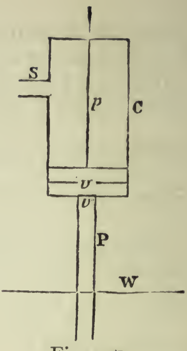


Fig. 117.

is shown in Fig. 117. W represents the level of the water below. P is a pipe not of greater length than 10 metres, and generally of much less length than that. C is a wider cylinder above, up and down which moves a piston *p*. This piston is moved by a lever not shown in the diagram. S is the spout leading the water off from C. In the piston is a valve *v* opening upwards. A second valve *v*′, also opening upwards, is at the top of the pipe P.

(*b*) Action.—Suppose the pump to be used for the first time. The piston is made to descend. The air in the part of C below the piston forces open the valve *v* in the piston, and passes through the piston. The valve *v*′ closes. The piston is then made to ascend. The air in the part of C above the piston closes the valve *v*, and is lifted out at S as the piston rises. There would be a vacuum in the part of C below the piston, but the air in P forces open valve *v*′ and passes into the part of the cylinder below the piston. Occupying thus a greater space than aforetime, this air exerts a less pressure on the surface of the water below than it did exercise, and the pressure of the air outside, no longer balanced, forces the water some distance up the pipe P.

The piston is again made to descend. Again the air in the part of C below the piston forces open the valve *v*, and passes through the piston. The valve *v*′ closes. The piston is then again made to ascend. The air in the part of C above again closes the valve *v*, and is lifted out at S as the piston rises. The air in P again forces open the valve *v*′, and passes into the lower part of C. The pressure upon the surface of the water being thus still further lessened, the pressure of the air outside forces the water yet farther up the pipe P.

These processes are repeated until water and not air lifts the valve at *v*′ and passes into the cylinder C. At the next descent of the piston *p*, water will force open the valve *v* and pass through the piston, and at the next ascent of the latter, water will be lifted up and driven out at the spout S.

As the pressure of the outside air has to sustain the water rising in the pipe P until it has passed through the opening guarded by the valve *v*′, the length of P must never be such that the air pressure could not support a column of water equal in length to P. The height of such a column of water, we have seen, would be about 10,260 mm. In practice the pipe P is generally much shorter than this.

[illegible]

(2) *The Force-Pump.*

(*a*) Construction.—On the model of the last, but the piston has no valve, and from the cylinder leads off a pipe with a valve opening upwards in it.

(*b*) Action.—When the piston descends, it pushes the air in the lower part of C up into the tube T, the valve *v* opening, the valve *v′* closing. When the piston ascends, the behaviour of the air in P, the water below, and the air outside is the same as it was with the common pump when its piston rose. These processes repeated, after a time water and not air lifts *v′* and passes into C. The next down-stroke of *p* forces water into T. The valve *v* always prevents any return of air or of water from T into C during the up-stroke of the piston.

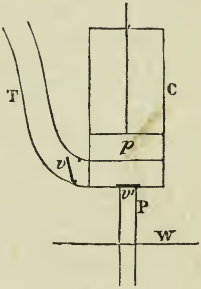


Fig. 118.

(3) *The Fire-Engine.*

(*a*) Construction. — Two force-pumps are used whose tubes communicate with a dome-roofed metal air chamber, in the upper part whereof is the air.

(*b*) Action.—As one pump descends and forces water into the air chamber, the other ascends, and causes the valve *v′* in Fig. 118 to open, and more water to rise from the cistern into C. The air in the air chamber is under great pressure from the continuous in-pumping of water, and is in a condition similar to that of a compressed spring. When any outlet from that air chamber is opened, the compressed air-spring forces the water out with great velocity along the hose.

D. Siphon.

The last three arrangements were for the purpose of raising water to a high level. The siphon is an arrangement for bringing water down to a lower level.

(1) *Construction.*—It is a bent tube, the open end of whose shorter limb is placed in the water to be removed, while the open end of the longer limb is free. The siphon must be filled with water

ere it will act. This is effected either by pouring water into it, placing fingers on the two open ends, immersing the end of the short limb in the water, and removing the fingers from the ends, or by sucking at the end of the longer limb until the tube is filled. Water then flows continuously out at this latter end.

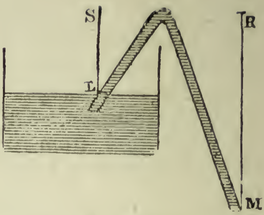


Fig. 119.

(2) *Action.*—Consider the topmost vertical layer of the water in the siphon. There is transmitted up the shorter leg to this layer the pressure of the atmosphere less the weight of the water in the short limb that has to be supported by the air. This weight is not the weight of all the water that is in the shorter limb. For, in the first place, so much of the water as is not above the level of the water outside is not supported by the air at all; and, in the second place, the sides of the short limb support some of the water weight. The air pressure really has to support the weight of a column of water of the same area as the siphon, and of vertical height = the vertical height from the level of the water to the topmost point in the siphon. Hence the pressure urging our topmost layer of water *away from* the vessel is

Air pressure – weight of column of water of height LS.

By similar reasoning, we arrive at the conclusion that the pressure urging our topmost layer of water *towards* the vessel is

Air pressure – weight of column of water of height MR.

The former pressure is greater than the latter, and the water is therefore forced from the vessel; and this happening with successive layers of water, there is a steady flow out at the open end of the longer limb. This flow will continue until the water in the vessel sinks below the open end of the short limb. It will be more rapid at first, and gradually diminish in speed; for whilst the height MR is constant, and therefore the air pressure – weight of column of water of height MR does not vary, the height LS is constantly increasing, and therefore the air pressure – weight of column of water of height LS is constantly diminishing, and the rate of flow must diminish with it.

E. Boyle's and Marriotte's Law.

It has been seen that a gas placed in any enclosed space diffuses over the whole of such space, no matter how small may be the quantity of gas concerned. If the dimensions of such an enclosed and enclosing space were to be made to contract, pressure would now be exerted upon the gas, and the volume of the gas would be lessened. To ascertain the exact connection between the volume occupied by gases and the pressure exerted upon them, was the object of Boyle and the object of Marriotte. These two observers, working independently, arrived at the same conclusion. Hence the 'law,' that is, the verbal expression of an observed order of events, is known sometimes as that of Boyle, sometimes as that of Marriotte. At present we are not considering the temperature of the gases studied; that temperature is for the present to be regarded as constant.

A bent glass tube is taken, with one limb shorter than the other (Fig. 120). This shorter limb is closed. A funnel is jointed on to the open end of the longer limb by a piece of india-rubber. At present the tube is full of air, and this air is the gas on which the experiment is to be tried. At present, also, this air in the bent tube is under the pressure of the atmosphere outside, as are all bodies at the surface of the earth. Let us assume that the barometer at the time stands at 760 mm., and the air-pressure is equivalent to that exerted by a column of mercury 760 mm. in height. Pour mercury in through the funnel until the bend of the tube is filled, and the air in the short leg is completely shut off from the air in the long leg (Fig. 121). That confined air is to be the especial portion of gas upon which the experiment is to be made. It is still only under the air pressure equivalent to 760 mm. of mercury. Its volume under that pressure is the volume of the short limb. Denote this by v.

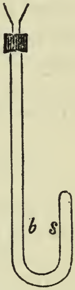


Fig. 120.

Under pressure = 760 mm., the gas occupies volume = v. Pour in more mercury *viâ* the funnel. The mercury rises more rapidly in the long open tube than in the short closed one, as in the latter its inflow is resisted by the imprisoned air. Continue to introduce mercury, until the fluid in the short limb is exactly half-way up the limb (Fig. 122). The volume of the air is now exactly half what it was under pressure 760 mm., or is $\frac{v}{2}$.

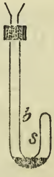


Fig. 121

Let us see what pressure it is under. First, there is the old air pressure = 760 mm. still acting and transmitted through all the mass of in-poured mercury. Second, there is the pressure due to that in-poured mercury, but not due to *all* that has been introduced. For the mercury in the bend of the tube exerts no pressure upon the air in the upper part of *s*. Nor does the mercury in the lower part of *s* exert any pressure on the air above it. It cannot press upwards; it is, indeed, pressing downwards with its full weight, and would fall, were it not for the mercury in the long limb. Just so much mercury in the long limb as there is of that fluid in the short limb, has to support the mercury in *s*, and cannot therefore press upon the air in *s* at all. What, then, is left to exert weight pressure upon the enclosed air? The column of mercury from *n* to *m* in the long limb, *i.e.* from a line level with the level of the quicksilver in *s*, up to the top of the liquid in *b*; a column of mercury, in brief, whose height = the difference of level of the fluid in the two limbs. Measuring this difference of level, it is found to be 760 mm. Hence the gas, occupying one-half its former volume, is found to be under a pressure of (1) the air = 760 mm. of mercury; (2) 760 mm. of mercury.

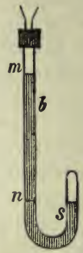


Fig. 122.

$$\text{Under pressure} = 760 \times 2, \text{ the gas occupies volume} = \frac{v}{2}.$$

If mercury is poured in until the volume of the air in *s* is reduced to one-third, the difference of level is found to be 1520 mm., or twice 760, and as the air pressure is still acting, and is still transmitted;

$$\text{under pressure} = 760 \times 3, \text{ the gas occupies volume} = \frac{v}{3}.$$

And, generally, it is established that under pressure of 760 × n, the gas occupies volume $= \frac{v}{n}$, *i.e.* the volume of a gas at constant temperature varies inversely as the pressure upon it. And this is Boyle's and Marriotte's law.

Example 76.—A Boyle's tube has a short limb whose contents are 60 c.c. What must be the difference of level of mercury in the two tubes, that the air may only occupy 25 c.c.? Barometer stands at 750 mm.

Given a Boyle's tube, with short limb whose contents = 60 c.c., and the barometer standing at 750 mm.

It is required to find the difference of level of mercury in the two limbs if the air occupies 25 cm.

The air occupies 60 c.c. under a pressure of 750 mm. mercury,

" " 1 " " " " 750×60,

" " 25 " " " " $\frac{\overset{30}{\cancel{750}} \times 60}{\cancel{25}} = 1800$;

$\therefore$ the difference of level $= 1800 - 750 = 1050$ mm.

Example 77.—A Boyle's tube has a short limb 2 decimetres in length. How much mercury must be poured into the tube that the level of that in the closed limb may be 2 centimetres from the top? Barometer at 768 mm.

Given a Boyle's tube whose short limb has a length of 20 cm., and the barometer standing at 768 mm.

It is required to find how much mercury must be poured in, so that the air in the short limb may occupy 2 cm. length.

18 cm. in the short limb are filled with mercury; $\therefore$ 18 cm. are required in the long limb to balance these, and total mercury *not* exerting pressure on air $= 36$ cm. $= 360$ mm.

The air occupies 20 cm. under a pressure of 768 mm. mercury,

" " 1 " " " " 768×20,

" " 2 " " " " $\frac{768 \times \overset{10}{\cancel{20}}}{\cancel{2}} = 7680$.

Total mercury required $= 7680 - 768 + 360 = 7272$ mm.

Example 78.—Into a Boyle's tube, whose short limb $= 3$ centimetres in length, and 1 square millimetre in section, water is poured until the air only occupies 10 millimetres of the short limb. What weight of water has been introduced? Barometer at 760 millimetres.

Given a Boyle's tube, whose short limb $= 30$ mm., into which water is poured until the air only occupies 10 mm.

It is required to find what weight of water has been introduced.

$20 + 20$ mm. of water are required to balance each other $= 40$ mm.

Air occupies 30 mm. under pressure of 760 mm. mercury.

" 1 " " 760×30.

" 10 " " $\frac{760 \times \overset{3}{\cancel{30}}}{\cancel{10}} = 2280$.

But a pressure = 760 mm. mercury is due to the atmosphere, and is merely transmitted ; ∴ 2280 − 760 = 1520 mm. are the amount of mercury required. The specific gravity of mercury = 13·5 ; ∴ 1520 × 13·5 = 20520 mm. water are required.

Total amount = 40 + 20520 = 20560 cubic mm.

Now the *weight* of water is required.

1 cubic centimetre = 1 gram,

$$20560 \text{ cubic mm.} = \frac{20560}{1000} \text{ cubic centimetres}$$

$$= 20{\cdot}56 \text{ grams.}$$

Example 79.—10 litres of gas are measured off at a pressure = 800 mm. of mercury. What volume will they occupy under normal pressure?

Given that 10 litres of gas are measured under 800 mm.

It is required to find what volume they occupy under normal pressure.

Normal pressure = 760 mm. mercury.

Under a pressure of 800 mm. the gas has a volume of 10 litres.
„ „ 1 „ „ 10×800
„ „ 760 „ „ $\frac{10 \times 800}{760}$

$$= \frac{200}{19}$$

$$= 10{\cdot}526 \text{ litres.}$$

Example 80.—To what additional pressure must 1 litre of gas under 760 mm. be subjected to reduce its volume to 3 decilitres?

Given that 1 litre of gas under 760 mm. is reduced to 3 decilitres.

It is required to find the pressure.

3 decilitres = $\frac{3}{10}$ litre.

Volume of gas is 10 decilitres under pressure of 760 mm.
„ „ 1 „ „ $760 \times 10.$
„ „ 3 „ „ $\frac{760 \times 10}{3}$

$$= \frac{7600}{3} = 2533{\cdot}\dot{3}\ ;$$

therefore the additional pressure is $2533{\cdot}\dot{3} - 760 = 1773{\cdot}\dot{3}$ mm.

Example 81.—5 litres of gas are measured over water, and the level of the water outside is 405 mm. higher than the level of the water inside the gas jar. What would be the volume of the gas under normal pressure? Barometer at 770. Specific gravity of mercury, 13·5.

Given that 5 litres of gas are measured over water, and that the water outside exerts a pressure = weight of column 405 mm. high, the barometer standing at 770 mm.

It is required to find the volume of the gas under 760 mm.

The gas is under a pressure of 770 mm. due to the atmosphere, and $\frac{405}{13·5} = 30$ mm. mercury; total, 800 mm. mercury.

Under pressure 800 volume of gas is 5 litres.
,, 1 ,, ,, 5×800.
,, 760 ,, ,, $\frac{5 \times 800}{760}$

$$= \frac{5 \times 20}{19} = \frac{100}{19} = 5\tfrac{5}{19}.$$

F. Air-Pumps.

Thus far we have considered air pressure and its effects, and the important generalization that the volume of a gas varies inversely as the pressure upon it. That which remains of pneumatics has to do with removal of air and diminution of air pressure, or with the converse of this process.

An air-pump is an arrangement for the removal of air from an enclosed space. Of such arrangement there are three principal forms.

(1) The simplest is represented in section in Fig. 12~~2~~3, *with Valveless Piston.*

(*a*) Construction.—

R represents the receiver or vessel to be exhausted of air.
T ,, the tube connecting this with C.
C ,, the cylinder or barrel of the pump.
p ,, the valveless piston.
v ,, a valve in the top of the cylinder C, opening upwards.

(*b*) Action.—The piston is made to descend. It forces the air in the lower part of C into R. Then passing the open end of T, it reaches the lowest part of C, and allows the air in T and R to pass *above* the piston into C. Then the piston is made to ascend. It lifts out, at v, all the air above it. Descending again, and passing the open end of T, more air flows from T and R above the piston into C, and the cylinder full of air is lifted out at v in the next ascent of p. It is clear that complete exhaustion cannot be effected by this air-pump.

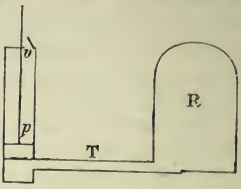


Fig. 123.

(2) *The Single-Barrelled Air-Pump with a Valve in the Piston.*

(*a*) Construction.—As the last, but the piston has a valve v' opening upwards, and the tube has a valve v'' where it joins the cylinder. This valve v'' also opens upwards.

(*b*) Action.—The action is that of the ordinary pump (page 150), only air is constantly being removed, and not water. Although a far more perfect exhaustion can be effected by this pump than by the last, two things tell against the exhaustion of air being complete even in this case.

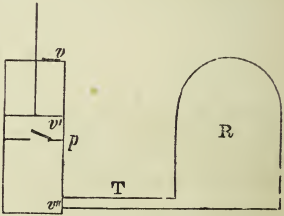


Fig. 124.

(i.) After a time the air becomes so rare that its elasticity is not sufficient to open the valve at v''.

(ii.) Suppose the capacity of the cylinder is exactly one-half that of the tube and receiver (T and R) taken together. At the first down-stroke of the piston, only the air in the lower part of cylinder passes through p, and at the first up-stroke only this air is removed. But at the second down-stroke one-half of the air in T and R passes through p, and at the second up-stroke this one-half is removed. Thus one-half of the air in T and R is left. The next down and up stroke remove one-half of this one-half, *i.e.* one-fourth. Thus one-fourth of the air in T and R is left. The next down and up stroke remove one-half of this one-fourth, or one-eighth. Thus one-eighth of the air in T and R is left. It will be seen therefore that some ever-diminishing fraction of air is left in T and R.

(3) *The Double-Barrelled Air-Pump.*

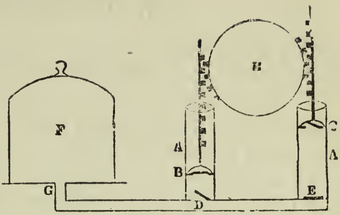


Fig. 125.

(*a*) and (*b*) Construction and Action.—As those of (2), but two single-barrelled pumps are used, and whilst one descends and allows air from the tube and receiver to pass through it, the other ascends and lifts out air that has already passed through it. In Fig. 125 the right hand piston is descending.

Example 82.—If the capacity of the tube and receiver of an air-pump be 3 litres, and the transverse section of the barrel be 10 square centimetres and its length 1 decimetre, find what volume of the original air will be removed by the second, third, and fourth strokes. The first only removes the air in cylinder.

Given that the capacity of the tube and receiver of an air-pump is 3 litres, and that the transverse section of the barrel is 10 sq. cm. and its length 10 cm.

It is required to find the volume of the original air removed respectively by the second, third, and fourth strokes.

1 c. dm. = 1 litre. ∴ 3000 c.c. = capacity of tube and receiver.
100 „ = „ barrel;

∴ $\frac{1}{30}$ contents is removed by each stroke of the piston.

Second stroke of the piston removes

$$\frac{1}{30} \times 3000 = 100, \text{ and leaves } 3000 - 100 = 2900.$$

Third stroke of the piston removes

$$\frac{1}{30} \times 2900 = \frac{290}{3}, \text{ and leaves } 2900 - \frac{290}{3} = \frac{8410}{3}.$$

Fourth stroke of the piston removes

$$\frac{1}{30} \times \frac{8410}{3} = \frac{841}{9}, \text{ and leaves } \frac{8410}{3} - \frac{841}{9} = \frac{24389}{9}.$$

Therefore after 2d stroke there remains = 2900.
„ 3d „ „ „ = $2803\frac{1}{3}$.
„ 4th „ „ „ = $2709\frac{8}{9}$.

G. The Condensing Syringe.

This is the exact converse of the air-pump as far as valvular arrangement and action are concerned.

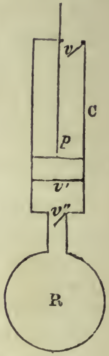


Fig. 126.

(1) *Construction.*—The valves at the top of the cylinder v, in the piston v', and at the beginning of the tube v'', open downwards.

(2) *Action.*—As p descends, v opens, and air enters C; v' closes, and air is forced into R, v'' opening downwards to allow its passage. As p ascends, v closes; the air between it and p passes through the latter, forcing down v', and v'' closes, keeping the air in R. The next descent of p forces into R the air that has passed through at the previous ascent. Hence the air in R grows constantly denser and denser.

Example 83.—In a condensing syringe, the capacity of whose receiver and cylinder are respectively 1 litre and 10 cubic centimetres, what will be the relative densities of the air before the syringe has been worked, and after 10 strokes?

Given that the capacity of the cylinder of a condensing syringe is 1 litre, and of the receiver 10 c.c.

It is required to find the relative densities of the air before the syringe has been worked, and after 10 strokes.

Let the density of the air before the syringe has been worked be represented by unity. 1 litre = 1 cubic decimetre = 1000 c.c.; ∴ each stroke of the piston increases the relative density of the air in the receiver by $\frac{1}{100}$ of original density. 10 strokes will therefore increase the density $10 \times \frac{1}{100} = \frac{1}{10}$. The relative densities, therefore, are 10 to 11.

EXERCISES.

1. A Boyle's tube, whose shorter limb is 20 cm. long, has 2580 mm. of mercury poured into it. To what height has the mercury risen in the shorter limb? Barometer 760 mm. unless otherwise specified.

2. What difference of level exists between the mercury in the two limbs of a Boyle's tube when the final volume of the air is $\frac{1}{6}$ the original volume of the air?

3. What is the length of the short limb of a Boyle's tube when, the difference of level being 114 cm., the air occupies 10 cm.?

4. 8 litres of gas are measured off at 760 mm. pressure. The pressure is increased to 800 mm. What alteration of volume occurs?

5. If the mercurial barometer show a rise of 27 mm., what is the rise in a barometer constructed of a fluid of sp. gr. 2?

6. Find the greatest length available of the suction pipe of a pump used to raise mercury, when the water barometer is at 1012 cm.

7. The receiver of an air-pump is six times as great in capacity as the barrel. Compare the densities of the air originally present, and of that remaining after two strokes.

8. If the receiver of an air-pump has a capacity of 2 litres, and after two strokes $1920\frac{4}{5}$ c.c. of the original air remain, find the capacity of the barrel.

PART II.

DYNAMICS

Thus far we have studied solids, liquids, and gases in conditions of equilibrium. The subject before us now is Dynamics, or the study of bodies in motion. In the earlier part of this book it was shown that two kinds of motion or change of place are known—viz. the movement of masses, or molar motion; and the movement of molecules, or molecular motion. Our divisions of this second branch of Natural Philosophy will be founded upon these two motions. First will be studied the movement of masses, or ordinary motion; then will follow the study of optics, or of the result of the wave-motion of ether; and the investigation of heat, or one of the results of the movement of molecules, will conclude our work.

MOVING BODIES.

First we shall study the terms used; then the laws of motion, and the formulæ that enable us to determine the speeds, times, ranges of bodies in movement; lastly, special cases of moving bodies.

CHAPTER XXI.

I. The Terms employed.

A. *Motion* is change of position. All motion is relative. Nothing upon the earth is absolutely at rest. Everything has the movements of the earth; but we shall only study movement in relation to the earth and things upon the earth. If a man is on board a vessel sailing parallel to a shore, he may (1) stand still on the vessel, (2) walk along the deck in the same direction as the ship is sailing, (3) walk along the deck in the opposite direction to that in which the ship is sailing. In case (1) his motion in respect to the ship is nothing; in respect to an object on the shore, it is a motion in the same direction as that wherein the vessel is moving. In case (2) his motion in respect to the ship is from aft to fore; in respect to an object on the shore, it is one in the same direction as that of the ship, but with a greater speed than that possessed by the latter. In case (3) his motion in respect to the ship is from fore to aft; in respect to an object on the shore, it is one in the opposite direction to that of the ship, or in the same, or is *nil*, according as he walks more rapidly, less rapidly, or at the same pace as the vessel sails.

B. *Space* is the distance traversed by any moving body, as measured from a certain starting-point, or the extent of its motion. It may be measured in feet, as in England, or metres, as on the Continent.

C. *Time* is the duration of the motion of any moving body. It is measured in seconds always; hence the foot or the metre is said to be the unit of space, the second the unit of time. If other units than these are employed, it is usually best to reduce the expression into feet or metres per second, especially in dealing with velocity. Thus 10 miles per hour would become $10 \times 1760 \times 3 \div 60 \times 60$ feet per second.

D. The speed at which a body moves is called its *velocity*. This velocity may be the same at different moments of time, and is then called *uniform;* or it may not be the same at different moments of time, and then it is called *variable*. Thus a train, when starting from a station and getting up speed, has a variable velocity; but when it is going at full speed, without let or hindrance, it has a uniform velocity, say of one mile in one minute. Velocity is measured by stating how far a body with such velocity, and such velocity only, would move in a unit of time, *i.e.* how many feet or metres it would pass over in one second.

If velocity is uniform, little difficulty presents itself. The velocity at any given moment being known, the velocity at all moments is known. The velocity is constant. But with variable velocities more difficulties present themselves. To denote the velocity or speed at any moment of a body moving with constantly-altering speed, we must suppose that for one second or for one unit of time its velocity does not change, but remains uniform during the lapse of the one second or of the one unit of time. Thus we might say of our train leaving the station, that as the last carriage quitted the platform, the train had a speed of one mile per hour. By this we should mean that if at the moment whereat the last carriage quitted the platform the engine ceased to work, and no friction of wheels against rails, and no air resistance were at work, the train would pass over one mile in the next hour. Or again, consider the case of a man running a 100 yards' race. His velocity is variable. At the end of the first 5 yards we may say roughly that his velocity is 4 yards per second, whereby we mean that if he were to run the whole distance at the speed he *now* has, without any increase, he would cover 4 yards each second, and make very poor time. But when he passes the winning-post, his velocity may be 12 or 15 yards per second, by which we should mean that could he continue at the speed he now has, he would cover 12 or 15 yards in every succeeding second.

Of this variable velocity, two kinds occur. The change of speed may be very irregular, increasing more in one second than in another, as in the case of a dog out for a walk, whose changes of velocity are by no means regular. Or the change of speed may be regular, just as great an increase or diminution taking place in one second as in its predecessor or successor, as in the case of a body falling earthwards under the attraction of gravitation. Hence we may have a non-uniform, *i.e.* variable *change* of velocity, and a uniform *change* of velocity. These variable and uniform *changes of velocity* must not be confounded with uniform and variable *velocities.* The changes of velocity can only occur under variable velocities. As this frequently-occurring phrase 'change of velocity' is cumbersome, it is replaced by the exceedingly important word 'acceleration.'

E. *Acceleration* is the change of velocity per unit of time that occurs in a unit of time. It may also therefore be uniform or variable, but henceforth we shall only deal with the uniform. Variable acceleration is not within our ken in this book, and all our attention will be concentrated upon uniform acceleration. Thus if any body is found to have a velocity of 40 metres per second at the beginning of a particular second, 30 metres per second at the beginning of the succeeding second, 20 metres per second at the beginning of the next second, its acceleration is − 10 metres per second.

Or if a body is found to have a velocity of $8\frac{1}{2}$ feet per second at the beginning of a particular second, 12 feet per second at the beginning of the succeeding second, $15\frac{1}{2}$ feet per second at the beginning of the next second, its acceleration will be $3\frac{1}{2}$ feet per second.

Acceleration, or the change of velocity per second in one second, may be either positive or negative—positive when the velocity increases, negative when the velocity diminishes.

As the terms space, time, velocity, acceleration will be constantly recurring in the succeeding pages, it will be well to have symbols for them.

Space will be denoted by the letter s.
Time ,, ,, ,, t.
Actual velocity at a given moment will be denoted by the letter v.
Acceleration will be denoted by the letter f.
To sum up these important terms :—

Term.	*Definition.*	*Unit.*	*Symbol.*
Space. . .	Distance from starting-point.	Foot or metre. .	s
Time. . .	Duration of movement.	Second. . . .	t
Actual velocity.	Speed of movement. .	Space traversed in one second. .	v
Acceleration. .	Change of velocity. .	Change of velocity per second that occurs in one second.	f

Example 84.—Express 17 kilometres 6 hectometres 4 decametres velocity per half-hour in the ordinary units.

$$\begin{aligned} 17 \text{ kilometres} &= 17000 \text{ m.} \\ 6 \text{ hectometres} &= 600 \text{ m.} \\ 4 \text{ decametres} &= 40 \text{ m.} \\ \hline & 17640 \text{ m.} \end{aligned}$$

$$\tfrac{1}{2} \text{ hour} = 30' = (30 \times 60)''.$$

$$\text{In } (30 \times 60)'' \text{ the velocity} = 17640 \text{ metres,}$$

$$\begin{aligned} \text{In } 1 \quad ,, \quad &= \frac{17640}{30 \times 60} \quad ,, \\ &= \frac{49}{5} \\ &= 5\frac{4}{5}. \end{aligned}$$

Example 85.—Express 10 miles 48 yards acceleration per 20 minutes in the ordinary units.

$$\begin{aligned} 10 \text{ miles} &= 10 \times 1760 \times 3 = 52800 \text{ feet} \\ 48 \text{ yards} &= 48 \times 3 \qquad\quad = 144 \text{ feet} \\ & \qquad\qquad\qquad\quad \overline{52944 \text{ feet}} \end{aligned}$$

$$20' = (20 \times 60)''.$$

In $(20 \times 60)''$ the velocity acquired $= 52944$ feet,

In $1''$ „ „ $= \dfrac{52944}{20 \times 60}$ feet

$$= \frac{4412}{5 \times 20} = \frac{1103}{5 \times 5} = 44\frac{3}{25}.$$

And the velocity per second acquired in $1''$ is the acceleration.

Exercises.

1. What acceleration has a body whose velocity at the end of one minute is 30 metres?

2. Express an acceleration of 9·8 metres per second in units of centimetres and half-hours.

CHAPTER XXII.

II. Laws of Motion.

Before proceeding to investigate the numerical relationships of space, time, velocity, acceleration, to one another, it will be well to master the three laws of motion.

A. If a piece of chalk be placed upon the ground, and no force of any kind, not even chemical action of air or water, or the many substances these carry, act upon it; if it be supposed to be left in the utterly impossible condition of entire loneliness, nothing ever acting upon it, it acting upon nothing, we are entitled to believe that the piece of chalk would remain in its position upon the ground without any change of place or condition whatever. This fact, turned into general expression, reads thus: that a body in a condition of relative rest does not tend to move from that condition unless force acts upon it.

Again, if a piece of chalk be thrown horizontally, and no force of any kind act upon it, we are entitled to believe that the piece of chalk would go on in a straight line with a uniform velocity for ever. In our practical experience we never even approximate to this condition of 'no force of any kind acting.' Air resistance and the attraction of the earth at least come into play, and the thrown chalk falls to the ground. But if we lessen these ever-acting forces, we find that as we lessen them the body thrown goes farther and farther. If, for example, a ball be thrown along a grass plat, it travels for a certain time. If it be thrown along a macadamized road, it travels a longer time. If it be thrown along a sheet of ice, it travels for yet a longer period. As we diminish friction in these cases, the farther and the longer does the thrown body travel. But in all these cases air resistance obstructs. Now, if a pendulum be set swinging in air, it swings backwards and forwards for a long time. If, however, it be made to swing in a vacuum, especially if friction at the point of suspension be reduced as much as possible, the pendulum will swing for very many hours. Hence it is inferred that, could we remove all extraneous forces entirely, a body set in motion by an impact force would move on in the line of its motion and with a uniform velocity for ever.

And thus the first law of motion, the sum of these two generalizations, states that—

A body at rest, or moving with uniform velocity, will remain at rest, or continue to move with that uniform velocity until acted upon by some other force.

This law is sometimes called the law of inertia, and if inertia be taken as the incapability of a body to change its state, either of rest or of motion, the name may be used.

B. If a man in an open carriage throw a ball vertically upwards as the carriage passes onwards, the ball does not fall in the road behind the vehicle, but into the man's hand. A circus rider, throwing up eggs and catching them, does not cast them into the air in front of him, but vertically upwards, as he would do were he standing on the ground. The eggs, although thrown vertically upwards from a particular position, come down to the rider's hand a few yards in advance of the place where he was when he cast them into the air. A stone let fall from the top of the mast of a sailing vessel falls down by the side of the mast on to the deck and not into the sea in the wake of the ship.

All these cases show that a body already in horizontal motion, when acted upon by a vertical force, undergoes a movement involving both horizontal and vertical change of place. And generally it may be said that when a force acts upon a body already under the action of another force, both forces have their full effect. This is true not alone of two, but of any number of forces. Every one has its full effect, and the resulting motion is a compound of all the several motions due to the several forces.

The parallelogram of forces is only a special statement of this principle. Thus, if at O is a particle that would be carried by a horizontal force from O to A in one second, and if whilst this horizontal force acts for the second, a vertical force that would carry the particle from O to B in one second also acts, we know from the parallelogram of forces that at the end of the second the particle would be at C. Now C is the position that would be occupied by the particle if (first) the horizontal force acted and bore it from O to A, and (second) the vertical force acted and bore it from A to C.

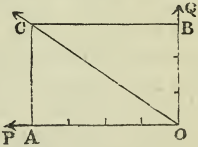


Fig. 127.

And thus the second law of motion, the verbal expression of the generalization derived from the above and from kindred observations, runs thus :—

If a body be acted upon by two or more forces, each force has its full effect.

C. When a bullet quits a gun, the gun 'kicks'—that is, as the bullet flies forward in one direction, the gun flies back in the opposite direction. And it is found that the amount of work done by the bullet and the gun are equal and opposite. We measure work by the product of the mass moved, and the space through which the mass moves. The small mass of the bullet moves through a great space. The huge mass of the gun moves through a little space. The 'action' of the bullet is attended with an equal and opposite 'reaction' of the gun.

A man leaps from the ground into the air. As he does so, he pushes the earth downwards. The amounts of work done are equal. The small mass of man moves through a considerable distance. The infinite mass of earth moves through an infinitely small distance. As the man under the attraction of gravitation descends through the same considerable distance as he previously ascended, the earth moves up through the infinitely small distance it previously descended. Any amount of jumping, it is satisfactory to learn, will not affect the earth's position. In each phase of the leap, it will be observed that the action of the man is accompanied by an equal reaction on the part of the earth.

Or again, a cannon ball strikes a target and falls to the ground. Apparently the enormous force of the moving cannon ball is lost. But could we calculate the amount of movement of the particles of the target where they have been driven more closely together by the ball, and also the amount of molecular movement producing heat resulting from the blow, we should find the force of the cannon ball not lost, but replaced by other forces. Its 'action' is equal to the 'reaction' of the target. Whenever a force of any kind seems to cease or to come into being, we always find its equivalent in the shape of some other force, which in comparison with the former 'action' (and only in comparison with it) we call the reaction. And thus the third law of motion, the verbal expression of this generalization, runs thus:—

Action and reaction are equal and opposite.

CHAPTER XXIII.

III. Formulæ for Moving Bodies.

A. General Formulæ.

It remains to discuss the various numerical relationships between the velocities, times, spaces, accelerations of moving bodies. These will be arranged under three heads :—

(1) When the body is moving under a uniform velocity.

(2) When the body is moving under a uniform acceleration.

(3) When the body is moving under a uniform velocity *and* a uniform acceleration.

(1) Bodies moving with Uniform Velocity. — A body after receiving a sudden blow or impact that gives it a certain velocity per second will, by the first law of motion, continue to move in the direction of the blow with that certain velocity for ever, unless other forces act upon it. Let us try to conceive such a case as that of a body moving with uniform velocity and acted upon by no other force, and then let us enquire as to the velocity it may have at any given moment, the space it may pass over in any given second, the space it may have passed over in any given time.

Once more, let s = number of feet or metres travelled,
t = number of seconds,
v = actual velocity at any given moment.

It will be seen that we need another letter to denote the velocity due to the sudden, once-acting force or impact. Let

V = velocity due to impact.

It must be carefully noted that V = number of feet or metres traversed in a second of time under the sudden, once-acting force.

(*a*) *To find the velocity of any given moment.*—As V is a constant, the velocity at any given moment will be V, or in symbols,

$$v = \mathrm{V},$$

i.e. the velocity or number of feet passed over per second at any moment = the initial velocity due to the impact.

(*b*) *To find the space passed over in the* t*th second.*—As V is a constant, the space passed over in any second is V.

$$S \text{ in } t\text{th second} = V,$$

i.e. the number of feet passed over in any second = the number of feet per second of the initial velocity.

(*c*) *To find the space passed over in* t *seconds.*—As V is the velocity per second, in t seconds the space passed over must be Vt.

$$S \text{ in } t \text{ seconds} = Vt,$$

i.e. the space traversed in t seconds = the product of the number of feet per second of the initial velocity and the number of seconds ot movement.

Example 86.—A body receives an impact that would urge it through 40 metres in 5 seconds. Find the ratio of the space passed over in a minute, the space passed over in the first half second of the second minute, and of the velocity after 10 seconds one to another. The formula here requred is $S = Vt$.

$$V = \frac{40}{5} = 8 \text{ metres per second.}$$

The velocity due to impact remains the same in whatever second it is taken. It is 8 metres in the first second, in the second second, and in the tenth second.

The space passed over therefore in a minute $= 8 \times 60$ metres.

" " in the first (or any) half-second of the second minute (or any minute) $= \frac{8}{2}$.

The velocity after $10'' = 8$ metres per second.

The ratio is $480 : \frac{8}{2} : 8$, or $120 : 1 : 2$.

(2) Bodies moving with Uniform Acceleration.—The same three inquiries present themselves ; as to the final velocity such a body may have, *i.e.* the velocity it may have at any given moment, as to the space it may pass over in any given second, as to the space it may have passed over in any given time.

Once more let s = number of feet or metres travelled,
t = " seconds,
v = actual or final velocity at any given moment,
f = acceleration or change of velocity per second in one second.

It may be said here, that unless the student fully comprehend the force of the word acceleration, and therefore the full significance of the symbol f, nothing that follows will be understandable. He is urgently advised to assure himself ere he read the next few paragraphs that he is master of f.

(*a*) *To find the velocity at any given moment.*

At the end of the 1st second $v=f$,
" " 2d " $v=2f$,
" " 3d " $v=3f$,
" " 4th " $v=4f$,
" " tth " $v=ft$,

i.e. the velocity at the end of any number of seconds = number of feet in the acceleration × number of seconds of movement.

Example 87.—With a uniform acceleration of 12 metres, find the velocity at the end of 5 seconds, and at the end of $10\frac{1}{6}$ seconds.

The formula here required is $v=ft$.
v is the unknown; $f=12$; $t=5''$ in the first case, and $10\frac{1}{6}''$ in the second;

$\therefore$ (i.) $v=12\times 5$,
$v=60$ metres per second at the end of 5 seconds.

And (ii.) $v=12\times 10\frac{1}{6}$
$=12\times\frac{61}{6}$
$=122$ metres per second at the end of $10\frac{1}{6}$ seconds.

(*b*) *To find the space passed over in the* t*th second.*

v at the beginning of 1st second $=0$,
" end " $=f$.

Average v per second during 1st second $=\dfrac{f}{2}$.

Space passed over in " $=\dfrac{f}{2}\times 1$.

Again, v at the beginning of 2d second $=f$,
" end " $=2f$.

Average v per second during 2d second $=\dfrac{3f}{2}$.

Space passed over in " $=\dfrac{3f}{2}=\dfrac{f}{2}\times 3$.

Again, v at the beginning of 3d second $=2f$,
" end " $=3f$.

Average v per second during 3d second $=\frac{5f}{2}$,

Space passed over in „ $=\frac{5f}{2}=\frac{f}{2}\times 5$.

Again, v at the beginning of 4th second $=3f$,
„ end „ $=4f$.

Average v per second during 4th second $=\frac{7f}{2}$,

Space passed over in „ $=\frac{7f}{2}=\frac{f}{2}\times 7$.

So that the space passed over in the 1st second $=\frac{f}{2}\times 1$.

„ „ „ 2d „ $=\frac{f}{2}\times 3$.

„ „ „ 3d „ $=\frac{f}{2}\times 5$.

„ „ „ 4th „ $=\frac{f}{2}\times 7$.

Now 1, 3, 5, 7 the multipliers of $\frac{f}{2}$ are the 1st, 2d, 3d, 4th odd numbers, and generally the space passed over in the tth second $=\frac{f}{2}\times t$th odd number. To obtain this tth odd number, t must be multiplied by 2, and 1 subtracted from the product. Thus the 4th odd number $=7$, *i.e.* $(4\times 2)-1$. Hence the formula reads:

$$\text{S in } t\text{th second}=\frac{f}{2}(2t-1),$$

i.e. the number of feet passed in a given second = half the number of feet in the acceleration × the tth odd number.

Example 88.—With a uniform acceleration of 24 feet, find the space travelled in the last second of the first minute, during the former half of the 6th second, and during the last fourth of the 7th second after starting.

The formula here required is $s=\frac{f}{2}(2t-1)$.

s is the unknown.
In the 1st case $f=24$, $t=60$.

In the 2d case half seconds are to be dealt with; $\therefore f=12, t=11$ (11th half-second).

In the 3d case, quarter seconds are to be dealt with; $\therefore f=6$, $t=28$ (28th quarter-second).

$$\text{(i.) } s=\frac{f}{2}(2t-1)=12\times 119=1428 \text{ feet.}$$

$$\text{(ii.) } s=\frac{f}{2}(2t-1)=6\times 21=126 \text{ feet.}$$

$$\text{(iii.) } s=\frac{f}{2}(2t-1)=3\times 55=165 \text{ feet.}$$

(*c*) *To find the space passed over in t seconds.*

Velocity at beginning of 1st second $=0$,
" end " " $=f$.

Average velocity per second for 1 second $=\frac{f}{2}$,

Space in 1 second $=\frac{f}{2}\times 1=\frac{f}{2}\times 1^2$.

Again, velocity at beginning of 1st second $=0$,
" end " 2d " $=2f$.
Average velocity per second for 2 seconds $=f$,

Space in 2 seconds $=f\times 2$, or $\frac{f}{2}\times 4=\frac{f}{2}\times 2^2$.

Again, velocity at beginning of 1st second $=0$,
" end " 3d " $=3f$.

Average velocity per second for 3 seconds $=\frac{3f}{2}$,

Space in 3 seconds $=\frac{9f}{2}$, or $\frac{f}{2}\times 9=\frac{f}{2}\times 3^2$.

Again, velocity at beginning of 1st second $=0$,
" " end " 4th " $=4f$.

Average velocity per second for 4 seconds $=2f$,

Space in 4 seconds $=2f\times 4$, or $\frac{4f}{2}\times 4$, or $\frac{f}{2}\times 16$, or $\frac{f}{2}\times 4^2$.

So that the space passed over in 1 second $= \frac{f}{2} \times 1^2$.
” ” ” ” 2 ” $= \frac{f}{2} \times 2^2$.
” ” ” ” 3 ” $= \frac{f}{2} \times 3^2$.
” ” ” ” 4 ” $= \frac{f}{2} \times 4^2$;

and generally the space passed over in t seconds $= \frac{f}{2} \times t^2$.

Hence the formula reads,

$$\text{S in } t'' = \frac{ft^2}{2},$$

i.e. the number of feet passed over in t seconds = half the number of feet in the acceleration × the square of the number of seconds.

Example 89.—What space will be passed over by a body moving with uniform acceleration of 8 metres per second in a quarter of a minute and in $3\frac{1}{2}$ seconds?

The formula here required is $s = \frac{ft^2}{2}$.

s is the unknown; $f = 8$; $t = 15''$ in the first case, and $3\frac{1}{2}''$ in the second.

(i.) $s = \frac{ft^2}{2} = 4 \times 15 \times 15 = 900$ metres in a quarter of a minute.

(ii.) $s = \frac{ft^2}{2} = 4 \times \frac{7}{2} \times \frac{7}{2} = 49$ metres in $3\frac{1}{2}''$.

Example 90.—Find the space passed over by a body moving with a uniform acceleration of $\frac{1}{25}$ metre in 5 seconds and in $12\frac{1}{2}$ seconds.

The formula here required is $s = \frac{ft^2}{2}$.

s is the unknown; $f = \frac{1}{25}$; $t = 5$ in the first case and $12\frac{1}{2}$ in the second.

(i.) $s = \frac{ft^2}{2} = \frac{1}{50} \times 5 \times 5 = \frac{1}{50} \times 25 = \frac{1}{2}$ metre in $5''$.

(ii.) $s = \frac{ft^2}{2} = \frac{1}{50} \times \frac{25}{2} \times \frac{25}{2} = \frac{25}{8} = 3\frac{1}{8}$ metres in $12\frac{1}{2}''$.

Example 91.—With what acceleration must a body move, that in $4\frac{1}{10}$ seconds it may pass over 168·1 feet?

The formula here required is $s=\frac{ft^2}{2}$.

$$s=168{\cdot}1 \text{ ft.};\ t=\frac{41}{10} \text{ seconds};\ f \text{ is the unknown.}$$

$$s=\frac{ft^2}{2};\ \therefore\ 168{\cdot}1=\frac{f}{2}\times\frac{41}{10}\times\frac{41}{10};$$

$$\therefore\ \frac{f}{2}=\frac{168{\cdot}1}{\frac{41}{10}\times\frac{41}{10}}=\frac{168{\cdot}1\times 10\times 10}{1681}=\frac{1681\times 10}{1681},$$

$$\frac{f}{2}=10;\ \therefore f=20.$$

so far.

We have now three formulæ dealing with bodies moving under uniform acceleration—(*a*) connecting v and t; (*b*) connecting s and t; (*c*) connecting s and t. But we have none connecting v and s. Such a fourth formula we build up from the first and third.

(*a*) $v=ft$; $\therefore$ squaring both sides of the equation, $v^2=f^2t^2$.

(*c*) $s=\frac{ft^2}{2}$; $\therefore$ multiplying both sides of equation by $2f$, $2fs=f^2t^2$;

$\therefore$ $v^2=2fs$, *i.e.* the square of the number of feet per second of the final velocity = twice the number of feet in the acceleration × the number of feet in the space traversed.

Example 92.—A body moving with acceleration 60 metres, is found to have a velocity = one-half the acceleration. Over what space has it passed?

The formula here required is $v^2=2fs$.

$$v=\frac{60}{2}=30,\ f=60,\ s \text{ is unknown.}$$

$$v^2=2fs;$$

$$\therefore\ s=\frac{v^2}{2f}=\frac{30\times 30}{2\times 60}=\frac{30}{4}=7\tfrac{1}{2} \text{ m.}$$

Example 93.—What will be the velocity of a body moving with an acceleration of 3 inches when the space it has described is 150 yards?

The formula here required is $v^2 = 2fs$.

$f = 3$ inches, $s = 150$ yards $= 150 \times 3 \times 12$ inches; v is unknown.

$$v^2 = 2fs,\ v^2 = 2 \times 3 \times 150 \times 3 \times 12 = 32400;$$
$$\therefore\ v = \sqrt{32400},$$
$$v = 180 \text{ inches}$$
$$= 15 \text{ feet.}$$

(3) BODIES MOVING WITH A UNIFORM INITIAL VELOCITY AND WITH A UNIFORM ACCELERATION.—We have now to consider the case of bodies set in motion by a sudden blow or impact, and at the same time under the influence of a constantly-acting force that causes acceleration. The most frequently-occurring cases of this double action are when bodies are thrown either upwards or downwards from or towards the earth. When a stone, for example, is flung vertically into the air, two forces are at work upon it—the upward fling, which is of the nature of an impact, and gives a uniform, unaltering velocity upwards; the earth's attraction working against the former and giving to the stone a uniform acceleration downwards. When a stone is thrown down from a height in a vertical direction, two forces also are at work—the downward fling, which is of the nature of an impact, and gives a uniform unaltering velocity downwards; the earth's attraction working with the former, and giving to the stone a uniform acceleration downwards.

In the former case the two forces are opposed; in the latter, they work together. In the latter case the formulæ used will be the sum of those under (1) and (2); in the former case the formulæ used will be the difference of those under (1) and (2); in both cases the algebraical sum.

(*a*) *To find the velocity at a given moment.*

As the actual or final velocity is the algebraical sum of the velocities due respectively to the initial constant velocity and the uniform acceleration,

$$v = V \pm ft,$$

i.e. the final velocity = the initial velocity ± the acceleration × the number of seconds during which the body has been moving.

Example 94.—A body has a velocity of $11\frac{1}{4}$ feet per second communicated to it, and also an acceleration of 3 inches per second in the opposite direction. Find its velocity at the end of $3''$.

The formula here required is $v = V - ft$.

$V = \frac{45}{4}$ feet, $f = 3$ inches or $\frac{1}{4}$ foot, v is unknown.

$$\begin{aligned} v = V - ft &= \tfrac{45}{4} - \tfrac{1}{4} \times 3 \\ &= \tfrac{45}{4} - \tfrac{3}{4} \\ &= \tfrac{42}{4} = 10\tfrac{1}{2} \text{ ft.} \end{aligned}$$

Example 95.—What must be the impact velocity given to a body that when an acceleration of $\frac{7}{3}$ metre per second works against the impact velocity, the final velocity at the end of 4 seconds may be 100 metres?

The formula here required is $v = V - ft$.

$v = 100$ metres, $f = \frac{7}{3}$, V is unknown.

$$\begin{aligned} v = V - ft,\ 100 &= V - \tfrac{7}{3} \times 4 = V - \tfrac{28}{3}, \\ 300 &= 3V - 28, \\ 328 &= 3V, \\ \tfrac{328}{3} &= V, \\ V &= 109\tfrac{1}{3}. \end{aligned}$$

(*b*) *To find the space passed over in the* t*th second.*

V is a constant, and the body passes over V feet in every second under the action of the original impact. It also passes, in every second, over a space $= \frac{f}{2} \times$ the odd number corresponding to the second. This is due to acceleration. The space traversed in a given second is the algebraical sum of the spaces due respectively to the initial force and to the acceleration.

$$s \text{ in } t\text{th second} = V \pm \frac{f}{2}(2t - 1),$$

i.e. the space passed over in the tth second = the number of feet of the initial velocity ± half the acceleration into the tth odd number.

Example 96.—Find the space passed over in the 5th second and in the first half of the 6th second by a body moving with uniform velocity of 100 feet per second, and acted upon at the same time by a force causing an acceleration in the same direction of 4 feet.

The formula here required is $s = V + \frac{f}{2}(2t - 1)$.

(i.) $V = 100, f = 4, t = 5,$

$$s = V + \frac{f}{2}(2t - 1) = 100 + 2 \times 9 = 118 \text{ ft.}$$

(ii.) Considering half-seconds, $V = 50, f = 2, t = 11$

$$s = V + \frac{f}{2}(2t - 1) = 50 + 1 \times 21 = 71 \text{ ft.}$$

Example 97.—In what second does a body with initial velocity of 75 metres per second, and an acceleration of 50 metres, working in the opposite direction, move over the same number of metres as are represented by the acceleration?

The formula here required is $s = V - \frac{f}{2}(2t - 1)$.

$$s = 50,\ V = 75,\ f = 50.$$

$$s = V - \frac{f}{2}(2t - 1);\ \therefore\ 50 = 75 - 25\ (2t - 1).$$

Dividing by 25, $2 = 3 - (2t - 1)$;

$$\therefore\ 2 = 3 - 2t + 1,$$

$$2 - 3 - 1 = -2t,$$

$$-2 = -2t;$$

$\therefore\ t = 1$, or the required second is the first.

(*c*) *To find the space passed over in* t *seconds.*

V is a constant, and the body passes over V feet in every second, and over Vt feet in t seconds. It also passes over $\frac{ft^2}{2}$ feet in t seconds under the action of the acceleration. The space traversed in a given number of seconds is the algebraical sum of the spaces due respectively to the initial force and to the acceleration.

$$s \text{ in } t \text{ seconds} = Vt \pm \frac{ft^2}{2},$$

i.e. space in t seconds = the product of the number of feet in the initial velocity and the number of seconds of movement, ± half the acceleration × the square of the number of seconds.

Attention must be especially called to the fact that in dealing with bodies moving under a force causing uniform velocity, *and* under a force causing uniform acceleration in the opposite direction, s stands for the distance of the body from the starting-point, not for the whole distance traversed. A body may be thrown vertically upwards, and after a time the action of the continued force of gravitation will bring the body to a stand-still, and then make it descend. In this case the whole space passed over is more than the distance of the body from the earth. This latter is represented by s. For example, when a body has been thrown upwards, and has come down to the earth again, $s = 0$, though the whole range is of course certain metres or feet. Another illustration may make yet plainer this important point.

Suppose a body thrown up with a velocity of 32 feet per second. (f due to gravitation also = 32). $s = \mathrm{V}t - \frac{ft^2}{2}$;

$$\therefore s \text{ in } 1'' = 32 \times 1 - (16 \times 1) = 16 \text{ feet},$$
$$s \text{ in } 2'' = 32 \times 2 - (16 \times 4) = 0 \text{ feet}.$$

It is manifest that in the 2 seconds the body has traversed several feet; yet $s = 0$, *i.e.* the body is on the ground again, and s stands for the distance from the starting-point, not the whole range.

Example 98.—Initial velocity = 3 metres, acceleration = 5 decimetres. Find space in 3 seconds when acceleration acts, first with, second against the initial velocity.

The formula here required is $s = \mathrm{V}t \pm \frac{ft^2}{2}$.

(i.) $\mathrm{V} = 3$ metres, $t = 3$, $f = \frac{1}{2}$ metre, s is unknown.

$$s = \mathrm{V}t + \frac{ft^2}{2} = (3 \times 3) + (\tfrac{1}{4} \times 3 \times 3)$$
$$= 9 + 2\tfrac{1}{4}$$
$$= 11\tfrac{1}{4} \text{ m}.$$

(ii.) $\mathrm{V} = 3$, $t = 3$, $f = -\frac{1}{2}$, s is unknown.

$$s = (3 \times 3) - (\tfrac{1}{4} \times 3 \times 3)$$
$$= 9 - 2\tfrac{1}{4}$$
$$= 6\tfrac{3}{4} \text{ m}.$$

Example 99.—If with an initial velocity of 8 feet, a body in 4 seconds passes forwards in the direction of this initial velocity, returns to the point of starting, and passes 32 feet beyond it, find the uniform acceleration that has been also at work during the 4 seconds.

The formula here required is $s = \mathrm{V}t - \frac{ft^2}{2}$.

s being the distance from the starting-point $= -32$ ft., $\mathrm{V} = 8$, $t = 4$.

$$s = \mathrm{V}t - \frac{ft^2}{2}; \therefore -32 = (8 \times 4) - \left(\frac{f}{2} \times 4 \times 4\right)$$
$$= 32 - 8f.$$

Dividing by -8, we have $4 = -4 + f$,

$$f = 8.$$

We have now three formulæ dealing with bodies moving under an impact *and* a uniform acceleration f; (a) connecting v and t; (b) connecting s and t; (c) connecting s and t. But we have none connecting v and s. Such a fourth formula we build up from (a) and (c).

(a) $v = V \pm ft$;

$\therefore$ squaring both sides of the equation, $v^2 = V^2 \pm 2Vft + f^2t^2$.

(c) $s = Vt \pm \dfrac{ft^2}{2}$;

$\therefore$ multiplying both sides of equation by $2f$, $2fs = 2Vft \pm f^2t^2$. Substitute for $2Vft + f^2t^2$ in the first derived equation $2fs$, its equivalent in the second derived equation, and we have

$$v^2 = V^2 \pm 2fs,$$

i.e. the final velocity squared = the initial velocity squared ± twice the acceleration × the number of feet passed over.

Example 100.—With initial velocity 70 metres, acceleration in the same direction 2 metres, the space traversed in a given time is 8 hectometres. What is the velocity at the end of that time?

The formula here required is $v^2 = V^2 + 2fs$.

$V = 70$ metres, $f = 2$, $s = 800$, v is unknown.

$$\begin{aligned} v^2 = V^2 + 2fs &= 70 \times 70 + 2 \times 2 \times 800 \\ &= 4900 + 3200 \\ &= 8100, \\ v &= \sqrt{8100}, \\ v &= 90 \text{ metres.} \end{aligned}$$

B. Graphic Representation of the Equations.

It is possible to illustrate certain of the results just obtained by geometrical diagrams. If a horizontal line be drawn whose length represents the lapse of time, and a vertical one whose length represents the velocity attained in that time, we have the following cases:—

(1) *Where a body moves with uniform velocity.*—The units of length in the line marked t may be taken as representing units of time or seconds. The vertical line marked V represents the velocity (constant). The space passed over is represented by the rectangle and $s = Vt$.

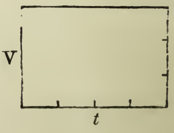


Fig. 128.

(2) *Where the body moves with a uniform*

acceleration.—The units of length in the horizontal line may be taken as representing seconds. The vertical lines represent the velocities at different times. Thus the velocity at the beginning of the time is 0, at the end of the 1st second, f; at the end of the 2d second, $2f$; at the end of the 7th second, $7f$; at the end of the tth second, tf. The space passed over is represented by the triangle. The area of a triangle = one-half the base × the height. The base here $= t$, the height $= tf$;

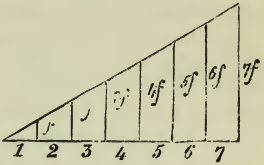


Fig. 129.

$$\therefore \text{ area} = \frac{t}{2} \times tf = \frac{ft^2}{2}, \text{ and } s = \frac{ft^2}{2}.$$

It should be noticed that the convenient phrase 'the space passed over is represented by the rectangle' or 'triangle,' as the case may be, if fully expanded, would read, 'the number of units of space traversed by the moving body is equal to the number of units of area in the rectangle or triangle.'

(3) *Where the body moves with a uniform velocity due to impact, and with a uniform acceleration.*—The units of length in the horizontal line may be taken as representing seconds. The line marked V represents the uniform velocity. The vertical lines marked f, $2f$, $3f$, etc., represent the velocities due to the acceleration at the end of successive seconds. The rectangle represents the space passed over as result of the uniform velocity. The triangle represents the space passed over as result of the acceleration. The whole figure represents the space passed over as result of both.

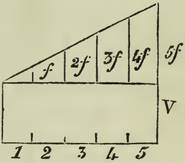


Fig. 130.

$$s = Vt + \frac{ft^2}{2}.$$

Fig. 131 shows the graphic representation of a case where the acceleration works *against* the initial velocity. The quadrilateral OABC represents the space passed over, and

$$s = Vt - \frac{ft^2}{2}.$$

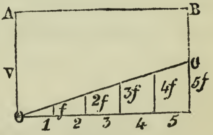


Fig. 131.

Example 101.—Draw a diagram to show the space passed over by a body moving (i.) with uniform velocity of $\frac{1}{2}$ metre per second for $6\frac{1}{2}$ seconds, (ii.) with uniform acceleration of $\frac{1}{2}$ metre per second for $6\frac{1}{2}$ seconds.

(i.) Let any unit represent 1″, and draw a horizontal line containing $6\frac{1}{2}$ of these units. Take any other unit to represent a metre, and draw a vertical line containing $\frac{1}{2}$ of this. The area of the rectangle contained by these two lines ($\frac{1}{2} \times 6\frac{1}{2}$) represents the space traversed.

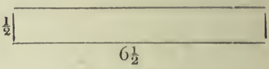


Fig. 132.

(ii.) Draw, as before, a horizontal line representing $6\frac{1}{2}$ units of time. At end of part representing first unit of time draw a vertical line to represent $\frac{1}{2}$ metre, at end of second another to represent 1 metre, and so on. The area of the triangle formed by joining the summits of the lines with the starting-point represents the velocity at the end of the $6\frac{1}{2}$ seconds.

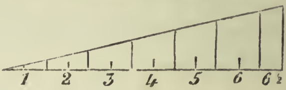


Fig. 133.

Example 102.—If the uniform velocity and the acceleration of the last question are both affecting a body, show graphically how to determine the space passed over in 4 seconds (i.) when the two work together, (ii.) when they are opposed.

(i.) Draw a horizontal line representing, as before, 4 units of time, and a vertical representing the uniform velocity of $\frac{1}{2}$ metre. Complete parallelogram, and then on upper line draw triangle representing the space passed over under the action of the force causing acceleration. TVAE represents the space passed over.

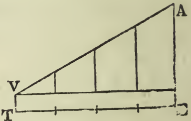


Fig. 134.

(ii.) Draw a triangle as before, but the parallelogram representing the space passed over due to the uniform velocity must be deducted from the triangle, as the acceleration acting for 4 seconds would make the body describe a greater space than it would under the uniform velocity. TAE represents space due to acceleration, TVDE space due to uniform velocity, BADC total space.

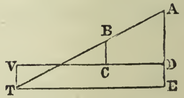


Fig. 135.

C. Summary.

Summing up the formulæ, we have—

Under head (1) only a uniform velocity affecting the body.

(*a*) $v = \mathrm{V}$, (*b*) s in tth second $= \mathrm{V}$, (*c*) s in t seconds $= \mathrm{V}t$.

Under head (2) only uniform acceleration affecting the body.

(*a*) $v=ft$, (*b*) s in tth second $=\frac{f}{2}(2t-1)$, (*c*) s in t seconds $=\frac{ft^2}{2}$, (*d*) $v^2=2fs$.

Under head (3) uniform velocity and uniform acceleration both affecting the body.

(*a*) $v=V\pm ft$, (*b*) s in tth second $=V\pm\frac{f}{2}(2t-1)$,

(*c*) s in t seconds $=Vt\pm\frac{ft^2}{2}$, (*d*) $v^2=V^2\pm 2fs$.

The student is earnestly advised, in dealing with any case of moving bodies, to ask himself, first, Under which of the three divisions, (1), (2), (3), does this case come? When this is decided, he should ask himself next, Which of the three or four formulæ under the particular head ought to be used in this special case?

EXERCISES.

✓ 1. Compare the velocities of a body moving with uniform velocity of 4 metres per second, and of a body moving with uniform acceleration of 1 metre per second at the end of 4 seconds and at the beginning of the 41st second. R

✓ 2. A body with uniform velocity passes in 8 seconds over 12 yards farther distance than it passed over in 5 seconds. Find velocity and space traversed in 5 and in 8 seconds.

✓ 3. A mass moves with uniform acceleration $\frac{1}{4}$ metre. Find velocity acquired in 8 seconds, space described in 8 seconds, space described in the 8th second.

✓ 4. With what acceleration does a mass move if, when its velocity is 300, the space described is 9000?

5. With uniform velocity of 8 metres a body moves for $5''$. Then an acceleration of 2 metres works in the opposite direction. Find the distance from the original starting-point at end of $10''$ from start, and at end of $\sqrt{56}+4$ seconds from the introduction of the acceleration. V

6. A mass moving under an impact and an acceleration in the same direction has at the end of $5''$ velocity $=200$, and has passed over 750 feet. Find impact velocity and acceleration.

CHAPTER XXIV.

IV. SPECIAL CASES.

THE general formulæ are supposed to be understood. Certain special cases of the application of them will now be considered. These are—A. Falling bodies; B. Descent down an inclined plane; C. Cases where the ratio of Pressure to Mass varies; D. Projectiles.

A. FALLING BODIES.—(1) When a body falls freely under the action of gravitation, we have a case coming under head (2); it is a case of a body moving under a uniform acceleration. Experiment has shown that the value of this acceleration is 9·8 metres per second, or 32·2 feet per second—that is, a body falling freely under the action of gravitation, acquires a velocity of 9·8 metres, or 32·2 feet in the first second, and in every succeeding second gains an additional velocity of 9·8 metres, or 32·2 feet. In making calculations about freely-falling bodies, the formulæ of (2) should be employed, with substitution of 9·8 metres or 32 feet (roughly) for f. (f, when a body is thus falling, is often written g. Thus when we say $g = 32$ or 9·8, we mean that the acceleration due to the earth's attraction upon a body allowed to fall freely is 32 feet or 9·8 metres.)

Example 103.—Through what space does a body falling under the action of gravitation pass in 10 seconds, and in the 10th second ($g = 9·8$ metres)?

The formulæ here required are—(1) $s = \frac{gt^2}{2}$,

and (2) $s = \frac{g}{2}(2t - 1)$.

$g = 9·8$ metres, $t = 10$.

(i.) $s = \frac{gt^2}{2} = 4·9 \times 10 \times 10 = 490$ metres.

(ii.) $s = \frac{g}{2}(2t - 1) = 4·9 \times 19 = 93·1$ metres.

Example 104.—A body falling under the action of gravitation has acquired a velocity of 208 feet per second. Through what space has it fallen? ($g = 32$ feet).

The formula required is $v^2 = 2gs$.
$g = 32$ feet, $v = 208$ feet, s is unknown.

$$v^2 = 2gs\,;\ \therefore\ 208 \times 208 = 2 \times 32 \times s,$$
$$s = \frac{208 \times 208}{2 \times 32} = 13 \times 52 = 676 \text{ feet.}$$

(2) Again, when bodies are thrown vertically downwards, with certain known velocities, gravitation acts as an acceleration in the same direction as the throw. Then the formulæ under (3) in last chapter come into play, with g replacing f, and the known velocity replacing V. The intermediate sign in these cases of vertically downward throw is +.

Example 105.—A body is thrown vertically downwards with a velocity = 4 decimetres. In what second will the body pass over 69 metres?

The formula required is $s = \mathrm{V} + \frac{g}{2}(2t - 1)$.

$$\mathrm{V} = \frac{4}{10} \text{ metre, } s = 69,\ t \text{ is unknown.}$$

$$s = \mathrm{V} + \frac{g}{2}(2t - 1)\,;\ \therefore\ 69 = \frac{4}{10} + 4{\cdot}9\,(2t - 1),$$
$$69 = \frac{4}{10} + 9{\cdot}8t - 4{\cdot}9,$$
$$690 = 4 + 98t - 49,$$
$$735 = 98t,$$
$$15 = 2t,$$
$$t = 7\tfrac{1}{2}\,;$$

$\therefore$ it will be during the $7\frac{1}{2}$th second, the second elapsing between the termination of the first half of the 7th second and the termination of the first half of the 8th.

Example 106.—With what initial velocity must a body be thrown vertically downwards, that when it has passed over 560 yards, its velocity may be 328 feet per second?

The formula required is $v^2 = \mathrm{V}^2 + 2gs$.

$$v = 328 \text{ feet, } g = 32,\ s = 560 \times 3,\ \mathrm{V} \text{ is unknown.}$$
$$v^2 = \mathrm{V}^2 + 2gs\ \therefore\ 328 \times 328 = \mathrm{V}^2 + 64 \times 560 \times 3,$$
$$107584 = \mathrm{V}^2 + 107520,$$
$$64 = \mathrm{V}^2,$$
$$\mathrm{V} = 8.$$

(3) When bodies are thrown vertically upwards with certain known velocities, gravitation acts as an acceleration in the opposite

direction to that of the throw. Then the formulæ under (3) in the last chapter are also used, with g replacing f, and the known velocities replacing V. The intermediate sign in these cases of vertically upward throw is $-$.

Example 107.—A body thrown vertically upwards has at the end of 6 seconds a velocity of 8·8 metres in a downward direction. Find its initial velocity.

The formula required is $v = V - gt$.

$$v = -8{\cdot}8 \text{ metres}, \ g = 9{\cdot}8, \ t = 6, \ V \text{ is unknown.}$$

$$v = V - gt; \ \therefore \ -8{\cdot}8 = V - 9{\cdot}8 \times 6,$$

$$-8{\cdot}8 = V - 58{\cdot}8,$$

$$V = 50.$$

Example 108.—A ball is thrown vertically upwards with a velocity of 80 feet. Where will it be at the end of 5 seconds?

The formula required is $s = Vt - \frac{gt^2}{2}$.

$$V = 80 \text{ feet}, \ g = 32, \ t = 5, \ s \text{ is unknown.}$$

$$s = Vt - \frac{gt^2}{2} = 80 \times 5 - 16 \times 5 \times 5$$

$$= 400 - 400,$$

$$s = 0.$$

That is, the ball will have risen till the vertical velocity is exhausted, and will have fallen again, touching the ground at the end of the 5 seconds.

(4) Two especially interesting cases present themselves in connection with bodies thrown vertically upwards against the action of gravitation. It is often interesting to know in such cases (a) how high the body will rise, (b) how long the body will rise.

(a) *To find the height to which a body will rise when thrown up with a vertical velocity represented by* V.

The question comes under (3), last chapter, formula (d), $v^2 = V^2 - 2fs$, $(f = g)$.

When the body is at its greatest height, v for an infinitely short space of time $= 0$;

$$\therefore \ 0 = V^2 - 2gs;$$

$$\therefore \ V^2 = 2gs;$$

$$\therefore \ s = \frac{V^2}{2g} = \frac{V^2}{64} \text{ or } \frac{V^2}{19{\cdot}6},$$

or the height to which a body will rise may be found by dividing the square of the number of feet or metres in the initial velocity by 64 or 19·6.

Example 109.—With what velocity must a stone be thrown vertically to rise to a height of $20\frac{20}{49}$ metres?

The formula required is $s=\frac{V^2}{2g}$.

$$s=20\frac{20}{49}\text{ metres}, g=9\cdot8, V\text{ is unknown.}$$

$$s=\frac{V^2}{2g};\ \therefore\ 20\frac{20}{49}=\frac{V^2}{19\cdot6},$$

$$\frac{1000}{49}=\frac{V^2}{19\cdot6},$$

$$19600=49V^2,$$

$$140=7V,$$

$$V=20.$$

(*b*) *To find the number of seconds during which a body will rise when thrown up with a vertical velocity represented by* V.

The question comes under (3), formula (*a*), $v=V-ft$, $(f=g)$.

At the end of the time of rising, v for an infinitely short space of time $=0$;

$$\therefore\ 0=V-gt;$$

$$\therefore\ V=gt;$$

$$\therefore\ t=\frac{V}{g}=\frac{V}{32}\text{ or }\frac{V}{9\cdot8},$$

or the number of seconds during which a body will rise may be found by dividing the number of feet or metres in the initial velocity by 32 or 9·8.

Example 110.—A body projected vertically upwards rises for 5 seconds. Find its initial velocity in feet.

The formula required is $V=gt$.

$$g=32\text{ feet}, t=5, V\text{ is unknown.}$$

$$V=gt;\ \therefore\ t=\frac{V}{g};\ \therefore\ 5=\frac{V}{32},$$

$$V=160.$$

B. Descent down an Inclined Plane.—If a body is sliding down an inclined plane without any friction whatever, yet the body does not fall so rapidly as if it be allowed to fall freely in a vertical direction. Part of the action of gravitation upon the body is neutralized by the resistance of the plane.

Consider the cases represented in (1), (2), (3), (4), (5), (6), Fig.

136. In (1) a ball is falling freely under the action of gravitation. In this case $f=g=9{\cdot}8$ metres or $32{\cdot}2$ feet.

In (2) a ball is resting on a perfectly horizontal surface. In this case f or the acceleration = 0.

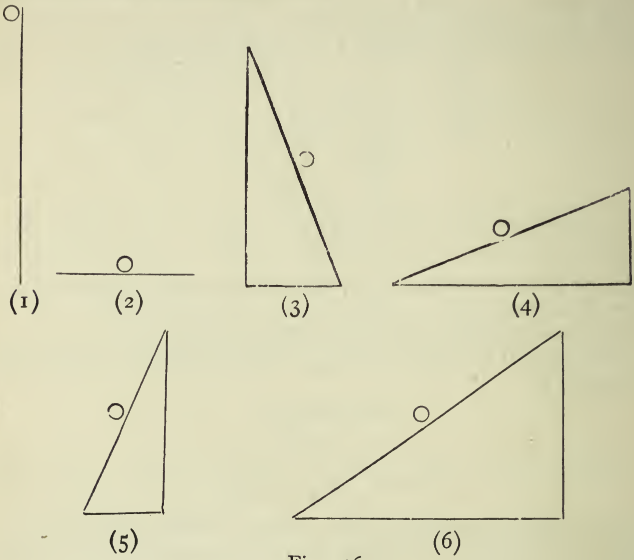


Fig. 136.

In (3), (4), (5), (6), balls are rolling down inclined planes, and f or the acceleration will in each case be less than g, but more than 0.

Consider (3) and (4). The ball represented in (3) will roll down with the greater acceleration as the plane in (3) is more nearly approximated to the vertical line than is the plane in (4), and the condition of the ball is approaching towards that of the ball in (1). The ball represented in (4) will roll down with the less acceleration as the plane in (4) is more nearly approximated to the horizontal line than is the plane in (3), and the condition of the ball is approaching towards that of the ball in (2). Now the lengths of the two planes in (3) and (4) are equal, but their heights are different. The height of the plane in (3) is greater, and the acceleration is greater in that case; the height of the plane in (4) is less, and the acceleration is less. Hence in bodies falling down inclined planes $f \propto h$, *i.e.* the acceleration varies directly as the height of the plane. Increase the height (without altering the length), and f is increased; diminish the height (without altering the length), f is diminished.

Further, the ball represented in (5) will roll down with the greater acceleration as the plane in (5) is more nearly approximated to the vertical line than is the plane in (6), and the condition of the ball is approaching towards that of the ball in (1). The ball represented in (6) will roll down with the less acceleration as the plane in (6) is more nearly approximated to the horizontal line than is the plane in (5), and the condition of the ball is approaching towards that of the ball in (2). Now the heights of the two planes in (5) and (6) are equal, but their lengths are different. The length of the plane in (5) is the less, and the acceleration in that case is the greater; the length of the plane in (6) is the greater, and the acceleration in that case is the less. Hence in bodies falling down inclined planes $f \propto \frac{1}{l}$, *i.e.* the acceleration varies inversely as the length of the plane. Increase the length (without altering the height), and f is lessened; diminish the length (without altering the height), and f increases.

As, therefore, a body falling freely has an acceleration represented by g, and the acceleration of a body falling down a smooth inclined plane varies as the height and inversely as the length, when a body does thus fall down an inclined plane,

$$f = \frac{gh}{l}.$$

In all cases of calculation in respect to bodies thus falling, find the height and the length of the plane. Multiply 9·8 or 32·2 by height ÷ length. Substitute this value of f in the particular equation selected from those under (2) in last chapter.

Example 111.—A body slides down a smooth inclined plane rising 2 in 7, and of length 34·3 metres. How long does it take to pass over the whole length?

Formula required is $s = \frac{ft^2}{2}$.

Plane rises 2 in 7, and 7 represents 34·3 metres;

$$\therefore \text{ 1 represents } \frac{34{\cdot}3}{7} = 4{\cdot}9,$$

$$\text{and 2} \quad ,, \quad 4{\cdot}9 \times 2 = 9{\cdot}8.$$

$$f = \frac{gh}{l}, \text{ and } s = \frac{ft^2}{2} = \frac{gh}{2l} \times t^2,$$

$$34{\cdot}3 = \frac{9{\cdot}8 \times 9{\cdot}8}{2 \times 34{\cdot}3} \times t^2,$$

$$t^2 = \frac{34{\cdot}3 \times 34{\cdot}3 \times 2}{9{\cdot}8 \times 9{\cdot}8},$$

$$t = \sqrt{\frac{34{\cdot}3 \times 34{\cdot}3 \times 2}{9{\cdot}8 \times 9{\cdot}8}},$$

$$t = \frac{34{\cdot}3\sqrt{2}}{9{\cdot}8},$$

$$t = \frac{4{\cdot}9\sqrt{2}}{1{\cdot}4},$$

$$t = \frac{{\cdot}7\sqrt{2}}{{\cdot}2},$$

$$t = \frac{7\sqrt{2}}{2},$$

$$t = \frac{7}{\sqrt{2}}.$$

Example 112.—Down a plane of angle 45° rolls a ball for 4 seconds. Find the length traversed.

Formula required is $s = \frac{ft^2}{2}$.

Length of plane $= s$; height of plane, $\angle$ being $45^\circ = \frac{s\sqrt{2}}{2}$; $t = 4$, $f = \frac{gh}{l}$.

$$s = \frac{ft^2}{2} = \frac{ght^2}{2l} = \frac{32 \times \frac{s\sqrt{2}}{2}}{2 \times s} \times 16 = 8\sqrt{2} \times 16$$
$$= 128\sqrt{2}.$$

In dealing with the velocity attained under these conditions of falling, trouble is often saved by remembering that the velocities of bodies falling the same *vertical* heights are always the same. Hence the velocity of a body that has rolled from the top of an inclined plane to the bottom is exactly the same as if the body had fallen a vertical distance = the height of the plane.

Example 113.—Show that with a plane of angle 30° and length 32 feet, the velocity of a body that has rolled from the top to the bottom is the same whether calculated by the formula $v^2 = 2fs$ (where s = the length of the plane), or $v^2 = 2gs$ (where s = the height of the plane).

In a plane with length 32 and angle 30°, the height is $\frac{32}{2}$ or 16.

$$\text{(i.)}\ v^2 = 2fs;\ \therefore\ v^2 = 2\frac{gh}{l} \times s = \frac{64 \times 16 \times 32}{32} = 64 \times 16.$$

$$\text{(ii.)}\ v^2 = 2gs = 64 \times 16.$$

The latter form is generally the better one to use, owing to its simplicity.

Example 114.—Find the velocity of a body that falls down an inclined plane of length 2 metres and base 1 metre 6 decimetres, when it has reached the bottom.

It is necessary to find the height of the plane. The plane is a right-angled triangle, the hypotenuse of which has a length of 20 decimetres, and the base = 16 decimetres; let x be the height.

$$20 \times 20 = 16 \times 16 + x^2,$$
$$400 - 256 = x^2,$$
$$\sqrt{144} = x;$$

∴ the height of the plane is 12 decimetres, and the velocity gained in rolling down the plane will equal the velocity gained by falling down a vertical height of 12 decimetres.

The formula required is $v^2 = 2gs$, s being the height of the plane.

$$v^2 = 2gs = 2 \times 9{\cdot}8 \times \frac{12}{10} = \frac{12}{5} \times 9{\cdot}8,$$
$$v = \sqrt{\frac{12}{5} \times 9{\cdot}8}.$$

As a derivative from the above we arrive at the important conclusion that the times of falling under the action of gravitation down any chord of a vertical circle from its highest point are constant, and equal the time of falling freely down the vertical diameter.

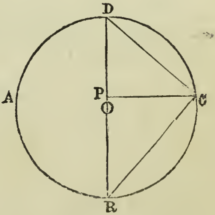


Fig. 137.

Let ADR be a circle whose highest point is D and vertical diameter is DR. Draw any other chord DC. It is required to prove that the time of falling under the attraction of gravitation from D to C is the same as the time of falling freely from D to R.

Join RC. From C draw CP perpendicular to DR.

A body falling from D to C falls down an inclined plane of height DP, length DC. When a body falls down an inclined plane,

$$f = \frac{gh}{l};$$

$$\therefore \text{ in this particular case } f = \frac{g \times \mathrm{DP}}{\mathrm{DC}},$$

$$\text{and } s = \frac{ft^2}{2};$$

$$\therefore \mathrm{DC} \text{ (the space described)} = \frac{g \times \mathrm{DP}}{2\mathrm{DC}} \times t^2. \qquad \text{(i.)}$$

Consider triangles DCR, DPC.

$\angle$ DCR and $\angle$ DPC are both right angles and are equal.

$\angle \mathrm{RDC} = \angle \mathrm{PDC}$.

$\angle \mathrm{CRD} = \angle \mathrm{PCD}$;

$\therefore$ triangles DCR, DPC are equiangular, are similar, and have their sides proportional;

$$\therefore \frac{\mathrm{DP}}{\mathrm{DC}} \text{ in } \triangle \mathrm{DPC} = \frac{\mathrm{DC}}{\mathrm{DR}} \text{ in } \triangle \mathrm{RDC}. \qquad \text{(ii.)}$$

Substitute in equation (i.) $\frac{\mathrm{DC}}{\mathrm{DR}}$ for its equal $\frac{\mathrm{DP}}{\mathrm{DC}}$, and we have

$$\mathrm{DC} = \frac{g \times \mathrm{DC}}{2\mathrm{DR}} \times t^2.$$

Dividing by DC, $1 = \frac{gt^2}{2\mathrm{DR}}$; $\therefore 2\mathrm{DR} = gt^2$;

$$\therefore t^2 = \frac{2\mathrm{DR}}{g}; \ \therefore t = \sqrt{\frac{2\mathrm{DR}}{g}}.$$

But 2DR and g are both constants; and further, if we consider the body as falling freely under the action of gravitation through the height DR,

$$s = \frac{gt^2}{2}; \ \therefore \mathrm{DR} = \frac{gt^2}{2}; \ \therefore 2\mathrm{DR} = gt^2;$$

$$\therefore t^2 = \frac{2\mathrm{DR}}{g}; \ \therefore t = \sqrt{\frac{2\mathrm{DR}}{g}},$$

i.e. the times of falling under the action of gravitation down any chord of a vertical circle from its highest point are constant, and equal the time of falling freely down the vertical diameter.

C. CASES WHERE THE RATIO OF PRESSURE TO MASS VARIES.—Thus far we have dealt with velocities and acceleration without, save partially in the case of falling bodies, considering especially the causes of the uniform velocities and uniform accelerations, or the masses of the bodies moved. We have now to take both these causes and masses into account. Let us agree to call the cause of motion Pressure, and to represent it by the letter P; to call the mass to be moved Mass, and to represent it by the letter M. It will be found that all the formulæ of (2) in the last chapter will come into use again, but *f*, or the acceleration, will need careful and special calculation in each case.

About the meaning of the word Pressure, or the force exerted, there will be little difficulty. The student may experience more difficulty in comprehending the meaning of the word Mass. Mass is quantity of matter. On no account must Mass, or quantity of matter, be confounded with weight or the force wherewith a quantity of matter is attracted earthwards. There is tendency to confusion because we habitually estimate Mass by weight. The same numbers represent both. None the less the two are quite distinct ideas. Suppose a mass of matter whose weight was 1 kilogram to be removed into such a position in space that its attraction towards the earth was exactly balanced by its attraction towards the sun and other heavenly bodies. Its weight would now be nothing. But its Mass would be exactly the same as it was when at the sea level. Or consider the case of pushing a kilogram weight along a perfectly smooth horizontal table. The pressure you exert does not overcome the weight of the kilogram. The resistance of the table is equal and opposite to that. Your pressure urges along the table a quantity of matter or a Mass. No weight is opposed to the force you employ, but a Mass is opposed to your force.

Let us now try to see how the value of *f* varies in relation to P and to M. If a cannon ball C be placed at one end of a

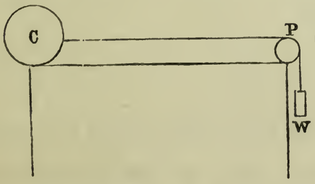


Fig. 138.

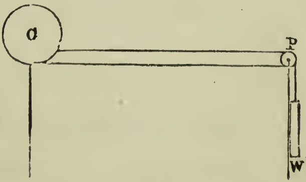


Fig. 139.

perfectly smooth horizontal table, and a string pass from it parallel to the surface of the table over a fixed pulley P, and have a small mass W attached to its other end, the cannon ball will begin to move along the table with a certain acceleration. If a precisely similar arrangement is made, save that the weight W′ is greater than W, the cannon ball will begin to move along the table with a greater acceleration. Now W and W′ are the two pressures in these two cases, and with the greater P, *i.e.* W′, f is greater, or $f \propto P$.

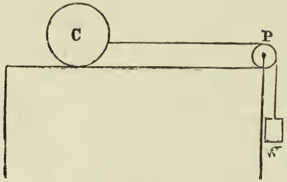


Fig. 140.

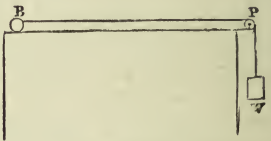


Fig. 141.

Again, with two similar arrangements wherein W is the same in each case, but the Mass to be moved is in one instance a cannon ball, in another a bullet, the acceleration would be less in the case of the cannon ball, greater in the case of the bullet. With the greater Mass f is less, or $f \propto \frac{1}{M}$, *i.e.* inversely as the Mass.

The greater the Pressure the greater the acceleration, or $f \propto P$.

The greater the Mass the less the acceleration, or $f \propto \frac{1}{M}$.

But when a quantity varies as two others (in this case as P and $\frac{1}{M}$), it varies as their product;

$$\therefore f \propto P \times \frac{1}{M} \propto \frac{P}{M}.$$

$f \propto \frac{P}{M}$ means therefore that the acceleration when a Pressure communicates motion to a Mass, varies as the Pressure and inversely as the Mass.

In the expression $f \propto \frac{P}{M}$ there are of course no concrete numbers. To obtain a calculable and workable form, consider the case of bodies falling freely under the attraction of gravitation.

Let a mass of one gram fall to the ground. What is the Pressure causing movement in this case? The weight of the body, *i.e.* the force wherewith it is attracted earthwards. The measure of that force is *one gram.* What is the Mass moved in this case? The quantity of matter in the gram weight. Our *measure* of that Mass is its weight, though Mass and weight are not identical. The Mass is denoted as a Mass of *one gram.* Here then with a body falling freely under the action of gravitation the same number 1 represents both P and M. $\frac{P}{M} = \frac{1}{1} = 1$. If a 20 gram weight is allowed to fall, the same number, 20, represents both P and M. $\frac{P}{M} = \frac{20}{20} = 1$. And so universally when bodies fall freely under the action of gravitation, the ratio of P to M is as 1 to 1, and $\frac{P}{M} =$ unity.

Now experiment has shown that under these last conditions $f = 32{\cdot}2$ feet or $9{\cdot}8$ metres. This value of f is denoted by g. When $\frac{P}{M} = 1$, $f = g$. Any other value of $\frac{P}{M}$ will give us $f = g \times$ that other value.

Thus, as when $\frac{P}{M} = 1, f = g$;

" $\frac{P}{M} = 10, f = 10g$;

" $\frac{P}{M} = \frac{1}{10}, f = \frac{1}{10}g$;

And generally $f = g \times \frac{P}{M}$, *i.e.* when a pressure P communicates motion to a mass M, the acceleration $f = 9{\cdot}8$ or $32{\cdot}2$ (g) $\times$ the Pressure ÷ the Mass.

In every case therefore where calculations have to be made as to velocity at a given time, or as to space travelled when the ratio of P to M is not 1, ask yourself, What is the Pressure causing movement, and what is the Mass moved? Then calculate from these and g the value of f, select the proper formula to use, and place for f in that formula the calculated value of $\frac{gP}{M}$.

Atwood's machine illustrates that which we have just been studying. It comprises a fixed pulley, revolving with as little friction as possible. Over this passes a flexible cord that carries a weight at either extremity. The weights are equal; but other weights can easily be placed upon one of them. These additional weights are in the form of bars, too long to pass through the ring F (moveable up and down the upright pillar), though that ring permits the weights attached to the string to pass through it. Thus the additional weight or weights can be removed at any particular point as the weighted weight descends, and then the space passed over in the next second is equal to the velocity gained. G is a stage also moveable up and down the upright pillar, and serving to stop the descending weight at any particular place; a pendulum beating seconds is also used. The action of this machine and its value in demonstrating the meaning of the formula $f=\frac{gP}{M}$ will be best understood by an example or two.

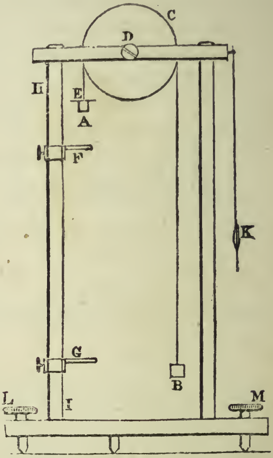


Fig. 142.

Suppose that each of the two equal weights = 3·45 decigrams, and that a weight = 1 centigram is placed upon one of them. Let us see what velocity the descending weight would have in 3 seconds. The pressure causing movement = 1 centigram. The mass to be moved = 6·9 decigrams or 69 centigrams + 1 centigram = 70 centigrams;

$$\therefore \frac{P}{M} = \frac{1}{70}; \therefore \frac{gP}{M} = \overset{1\cdot4}{\cancel{9\cdot8}} \times \frac{1}{\underset{10}{\cancel{70}}} = \cdot14 \text{ metre} = 14 \text{ centimetres } (f).$$

Now $v=ft$; $\therefore$ v = 14 centimetres × 3 = 42 centimetres = 4 decimetres 2 centimetres.

And if the ring were arranged so as to take off the extra weight of 1 centigram at the end of the 3 seconds, the descending weight would thereafter pass over 4 decimetres 2 centimetres in each succeeding second. There would be no increase of velocity after the 1 centigram was removed, as the two other weights are equal, and balance one the other, as far as gravitation is concerned.

The question may be asked, Where should we place the ring so as to take off the extra weight of 1 centigram at the end of the three seconds? That can be calculated thus:

$$s = \frac{ft^2}{2}. \quad \text{But } f \text{ in this case} = 14 \text{ centimetres};$$

$$\therefore s = (7 \times 9) \text{ centimetres} = 63 \text{ centimetres} = 6 \text{ decimetres } 3 \text{ centimetres},$$

and the ring must be placed 6 decimetres 3 centimetres below the level of the top of the descending weight just as it begins to move.

Example 115.—An Atwood's machine has weights, each of 6 decigrams; $\frac{1}{2}$ decigram is placed upon one of them. Where must the ring be placed to take off the extra weight at the end of 5 seconds, and over what space would the weights pass in the next 2 seconds?

$$f = \frac{gP}{M}, \ P = \tfrac{1}{2}, \ M = 12\tfrac{1}{2}, \ g = 9{\cdot}8;$$

$$\therefore f = \frac{9{\cdot}8 \times \frac{1}{2}}{12\frac{1}{2}} = \frac{9{\cdot}8}{25}.$$

(i.) The formula required is $s = \frac{ft^2}{2}$,

$$s = \frac{4{\cdot}9}{25} \times 25,$$

$$s = 4{\cdot}9 \text{ metres.}$$

When the bar is taken off, the weights have a certain velocity.

$$v^2 = 2fs \text{ and } s = 4{\cdot}9;$$

$$\therefore v^2 = \frac{19{\cdot}6}{25} \times 4{\cdot}9, \text{ and } v = \frac{9{\cdot}8}{5}.$$

(ii.) Formula required is $s = Vt + \frac{ft^2}{2}$

$$= \frac{9{\cdot}8}{5} \times 2 + \frac{4{\cdot}9}{25} \times 4 = \frac{19{\cdot}6}{5} + \frac{19{\cdot}6}{25} = \frac{117{\cdot}6}{25} \text{ metres.}$$

Example 116.—What weight must be placed on one of two weights in an Atwood's machine, if each of the original weights

$= 7\frac{1}{2}$ grams, in order that when the velocity is 6 feet per second the space passed over may have been 4 feet?

$f = \frac{gP}{M}$; P is unknown, and may be represented by x; M is $7\frac{1}{2} \times 2 + x = 15 + x$; $\quad \therefore f = \frac{x}{15 + x} \times 32.$

The formula required is $v^2 = 2fs$;

$$\text{whence } s = \frac{v^2}{2f}. \quad \text{But } s = 4,\ v = 6,\ 2f = \frac{2x \times 32}{15 + x},$$

$$s = \frac{v^2}{2f}; \therefore 4 = \frac{6 \times 6}{\frac{2x \times 32}{15 + x}}; \therefore 4 = \frac{36 \times (15 + x)}{2x \times 32},$$

$$1 = \frac{9(15 + x)}{64x},$$

$$64x = 135 + 9x,$$

$$55x = 135,$$

$$11x = 27,$$

$$x = \frac{27}{11} = 2\frac{5}{11} \text{ grams.}$$

Example 117.—A mass of 10 grams is drawn along a horizontal and smooth table by a mass of 2 grams. It reaches the end of the table in 2 seconds. Find the length of the table.

$f = \frac{gP}{M}$, $P = 2$ grams, $M = 10 + 2$ grams, $g = 9{\cdot}8$ metres;

$$\therefore f = \frac{2 \times 9{\cdot}8}{12} = \frac{4{\cdot}9}{3}.$$

Formula required is $s = \frac{ft^2}{2}$, where $t = 2$,

$$s = \frac{4{\cdot}9}{6} \times 4 = \frac{4{\cdot}9}{3} \times 2$$

$$= \frac{9{\cdot}8}{3}$$

$$= 3{\cdot}2\dot{6} \text{ metres.}$$

Example 118.—In a like experiment to the one in the preceding question, what Mass will, under the action of a weight of $\frac{1}{2}$ gram, acquire a velocity in 3 seconds that would carry it over $\frac{8}{67}$ foot in the next $\frac{1}{4}$ second?

$f = \frac{gP}{M}$; $P = \frac{1}{2}$; M is represented by x; $f = \frac{\frac{1}{2} \times 32}{\frac{1}{2} + x} = \frac{32}{1 + 2x}$;

v given is $\frac{8}{67}$ in $\frac{1}{4}$ second, or $\frac{32}{67}$ per second.

Formula required is $v=ft$,

$$\frac{32}{67}=\frac{32}{1+2x}\times 3,$$

$$\frac{1}{67}=\frac{3}{1+2x},$$

$$1+2x=201,$$

$$2x=200,$$

$$x=100 \text{ grams.}$$

Example 119.—A pressure = 1 gram acts upon a Mass of 4 grams. Find the space passed over in the third second.

$f=\frac{gP}{M}$; P = 1 gram; M = 4 grams; $g=9{\cdot}8$; $\therefore\ f=\frac{1\times 9{\cdot}8}{4}$ $=2{\cdot}45$.

Formula required is $s=\frac{f}{2}(2t-1)$

$$=1{\cdot}225\times 5$$

$$=6{\cdot}125 \text{ metres.}$$

D. PROJECTILES.—Our last study under the head of moving bodies will be that of projectiles. Here the second law of motion comes into play. When a shot is discharged from a gun, two forces in the main determine its course—(1) the force driving it from the gun in a horizontal or slanting direction; (2) the attraction of the earth. Each of these has its full effect, and the student who has mastered the meaning of the formulæ for bodies moving under uniform velocity, the meaning of the formulæ for falling bodies and the parallelogram of forces, will have no difficulty here. Strictly speaking, in discussing the formulæ under pp. 187–189 we were dealing with projectiles. In that discussion we considered only bodies thrown vertically up or vertically down. In this place, therefore, we shall only consider bodies thrown at some angle between 0° and 180° with the vertical line.

Suppose a stone is thrown horizontally with a velocity of 7·35 metres per second, and it is required to find its distance from the point of projection at the end of two seconds. Let O be the point of projection. Measure from O along a horizontal line 2 units of length, each of which is to represent 7·35 metres. OB represents (7·35 × 2) metres = 14·7.

The stone would travel 14·7 metres horizontally in 4 seconds.

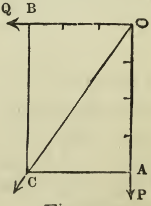


Fig. 143.

From O draw a vertical line, and measure off along this the number of metres that the stone would fall under the action of gravitation.

$$s=\frac{gt^2}{2}=4{\cdot}9\times 4=19{\cdot}6 \text{ metres.}$$

The stone would, by the parallelogram of forces, pass, not to A nor to B, but to C. It is required to find the distance from O to C.

$$OC^2=BC^2+OB^2=(19{\cdot}6)^2+(14{\cdot}7)^2=(4\times 4{\cdot}9)^2+(3\times 4{\cdot}9)^2$$
$$=(16+9)\,(4{\cdot}9)^2=25\times 4{\cdot}9^2.$$
$$OC=\sqrt{25\times 4{\cdot}9^2}=5\times 4{\cdot}9=24{\cdot}5 \text{ metres.}$$

The stone would not pass directly along the line OC, as the horizontal velocity is uniform from the first instant of motion, but the vertical velocity begins at nothing and grows greater. The actual line described is a parabola.

Example 120.—A stone is thrown with a horizontal velocity of $\sqrt{11529}$ feet per second from the top of a tower. In 4 seconds it is on the ground. Find the height of the tower, and the distance of the stone from its summit.

The horizontal distance travelled by the stone is found by the formula $s=vt$;

$$\therefore s=\sqrt{11529}\times 4.$$

The height of the tower is the space through which a stone would fall in 4 seconds under the action of gravity; the formula is $s=\frac{gt^2}{2}$;

$$\therefore s=16\times 16=256.$$

The distance of the stone from the top of the tower is found by drawing a parallelogram, of which the length is $4\sqrt{11529}$, and the height is 256; the diagonal is the distance of the stone from the top of the tower. Let this diagonal be represented by x.

Then $x^2=256^2+(4\sqrt{11529})^2$

$$=65536+184464=250000,$$
$$x=\sqrt{250000}=500 \text{ feet.}$$

Example 121.—With what velocity must a body be thrown into the air at an angle of 30° with the ground that in ten seconds it may strike the earth? How far from the point of projection will the ball be at the end of the 10 seconds?

Decompose the initial velocity into its horizontal and vertical components.

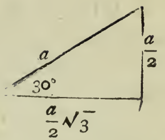


Fig. 144.

With v represented by $\frac{a}{2}\sqrt{3}$, the body will travel horizontally, and the space thus travelled over must be calculated.

With v represented by $\frac{a}{2}$, the body will travel vertically upwards until the velocity is exhausted.

The time of rise equals time of fall; whole time is ten seconds; therefore time of rise $= 5$ seconds, and time of fall $= 5$ seconds.

Formula required to find distance through which body will fall in 5 seconds is $s = \frac{gt^2}{2}$; $\therefore s = 16 \times 25 = 400$; therefore the body falls through 400 feet. But the velocity acquired in falling through 400 feet will equal velocity exhausted in rising 400 feet. Formula required to find this velocity is

$$v^2 = 2gs,\ v^2 = 64 \times 400,\ v = \sqrt{25600} = 160.$$

Therefore $\frac{a}{2}$ represents a velocity of 160, and a the initial velocity $= 320$.

Also $\frac{a}{2}\sqrt{3}$, the horizontal velocity, $= 160\sqrt{3}$.

This velocity continues for 10 seconds; formula required to find distance from point of projection is

$$s = Vt = 10 \times 160\sqrt{3} = 1600\sqrt{3}.$$

$$g = 32 \text{ feet or } 9{\cdot}8 \text{ metres.}$$

EXERCISES.

1. A body falls freely under action of gravitation. Compare the number of feet passed over in $10''$ and the velocity in metres attained in $163\frac{13}{49}''$.

2. In what second does a body falling freely traverse 44 metres 1 decimetre?

3. A body is thrown vertically upwards with velocity 98 metres. To what height does it rise?

4. A stone thrown vertically upwards reaches the ground in $10''$. With what velocity was it thrown up, and how high did it rise?

5. Find the time of descent down an inclined plane whose height is $32\sqrt{3}$ feet, angle 60°; also the velocity acquired.

6. What is the velocity of a body that falls down an inclined plane rising 50 $\sqrt{2}$ feet in 100, if the length of the plane is 192 $\sqrt{2}$?

7. A mass of 4 moves itself and a mass of 12. Find the space in feet and in metres described in the fourth second.

8. What mass must be placed on one of two equal masses of 4·4 grams, that when the space traversed is 18 metres, the velocity acquired may be 6 metres?

9. A ball thrown horizontally with velocity = 96 feet falls under action of gravitation. Find its position at end of 3″.

10. A stone thrown from the top of a tower with horizontal velocity = 490 metres reaches the ground in 2″. Find height of tower and distance of stone from summit when it reaches the ground.

OPTICS.

THE movement of masses has been dealt with in the preceding pages. The investigation of wave movement or undulatory movement of ether leads us to Optics, or the study of light.

Optics will be investigated under the following heads:—the nature of light; the chief terms employed; absorption of light; reflection of light; refraction of light.

CHAPTER XXV.

I. The Nature of Light.

IN the old days, light, like everything else, was a form of matter. As the phenomena of heat were due to a fluid, caloric, those of electricity to a fluid, those of life to a soul; so the phenomena of light were due to actual material particles thrown off by luminous bodies, and sent either directly from them or indirectly from them, after reflection from other bodies. Of later years, the various great physical forces have grown gradually to be regarded as modes of motion rather than of matter. Heat, as will be seen hereafter, is a mode of motion of molecules; electricity and magnetism and life are probably modes of motion of molecules; and light is due to the motion of a weightless, universally-diffused medium called ether.

A. The Newtonian theory of light is dead. That theory held that actual particles of matter were cast off from luminous bodies, and passing directly to animal eyes, or indirectly, after reflection from visible bodies, caused the sensation of sight. This was called the corpuscular theory, from *corpuscula*, little bodies.

B. To-day the undulatory theory is accepted. Throughout space is ether, a weightless medium. Luminous bodies throw this medium into waves, as the wind makes waves in the sea. These waves, some travelling for many thousands of years, at the rate of about 180,000,000 miles per second, fall at last upon the retina of the eye, or are reflected indirectly upon that retina by visible bodies. In either case, these waves of ether, whose shore is the nervous layer or retina of the eye, impinging upon that shore set in motion other molecules of a nerve nature; these in their turn probably transfer the motion to molecules in the brain, and the result is the consciousness of light. And this theory is the undulatory or wave theory of light.

II. Terms employed in the Study of Optics.

A. The smallest conceivable portion of light is called a *ray*. An ordinary ray of sunlight consists of infinite numbers of these rays. The conception of the theoretical ray of light is as imaginary as that of a line in geometry. But the one conception is as useful as the other. A ray will therefore mean to us the smallest conceivable portion of light, and will be represented by a straight line.

B. Collections of rays are called *pencils*. The ray of ordinary sunlight would be really a pencil, or collection of rays. As all the rays in such a pencil of sunlight are parallel each to the other, the pencil is called a *parallel pencil*. Fig. 145 represents such a parallel pencil. Very frequently, however, rays of light, instead of running all parallel one to another, diverge from or converge towards each other. The pencils whereof such rays make part are then called *divergent pencils* or *convergent pencils* respectively. The lines passing from O (Fig. 146) represent a divergent pencil; Fig. 147 represents a convergent pencil.

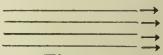


Fig. 145.

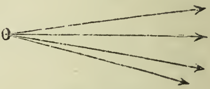


Fig. 146.

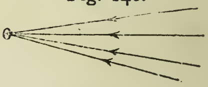


Fig. 147.

C. The point O (Fig. 146), from which the divergent rays pass, and the point O (Fig. 147), to which the convergent rays pass, are called the *foci* of the two pencils. A *focus*, therefore, is the point from or towards which rays of light pass.

D. When light falls upon a body, three things happen to the light. Some of it passes into the body, is completely *absorbed* thereby, and is seen no more; some of the light is thrown off from the body, or reflected from it; some of the light passes through the body generally, and in doing so is bent out of its original course, or *refracted*. Our study of optics will henceforth fall under the three heads of absorption of light, reflection of light, refraction of light.

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III. Absorption of Light.

When light falls upon a body, some of the light is absorbed unless the body is perfectly transparent, and as yet no perfectly transparent body is known. Upon the portions of light absorbed and not absorbed respectively by the body, depends its colour. Many years ago Isaac Newton showed that white light, as that of the sun, is not simple but compound. Allowing such white light to pass through a glass prism, he found that the light that had been simply white when entering the prism at one side, was, on its emergence from the other, broken up into seven colours, each graduating into its neighbour on either hand. The seven colours resulting from this decomposition of white light are violet, indigo, blue, green, yellow, orange, red.

Now when white light falls upon any body, either none of these colours is absorbed and the body is white, or all are absorbed and the body is black or has no colour, or one or more is or are absorbed and the body is of a colour dependent upon the rays *not* absorbed but reflected or refracted. For example, when light falls upon the coat of a soldier, the coat absorbs the violet, indigo, blue, green, yellow, and orange rays, and only reflects red to the admiring eyes of the servant-girl. When light falls upon certain spring flowers, the red, orange, yellow, green, blue, and indigo are absorbed, and only such a colour is reflected as leads us to name the flower the violet. The colour therefore of any structure depends upon the kind of light it does not absorb, but either reflects from it or allows to pass through its substance.

CHAPTER XXVI.

IV. Reflection of Light.

Many bodies when light falls upon them throw some of that light back from their surfaces into the medium through which the light has passed to them. If light fall, for example, upon a looking-glass or upon a burnished frying-pan, a great deal of it is thrown back or reflected, and bodies like the metals that have this power of casting light back when it has fallen upon them are called reflecting bodies. No bodies yet known are perfect reflectors, *i.e.* reflect all the light that falls upon them, but the metals when polished are amongst the best reflectors known. Indeed, in a looking-glass, the metal mercury behind the glass is the real reflector.

A. Terms used.

5 In the study of the reflection of light five new terms must be introduced. The rays that fall upon the reflecting surface are called *incident rays* (from *incido*, I fall upon). The rays that are thrown back from the reflecting surface are called the *reflected rays* (from *reflecto*, I bend back). A line drawn at right angles

to any reflecting surface, *i.e.* a perpendicular to the surface, is called the *normal* to that surface. The angle contained between the incident ray and the normal is called the *angle of incidence.* The angle contained between the reflected ray and the normal is called the *angle of reflection.*

So far

B. Laws of Reflection.

Experimentally, two very simple laws have been discovered that make the study of the reflection of light not very difficult of comprehension.

Suppose M represents a plane reflecting surface or mirror, and I represents an incident ray passing from a luminous point to the mirror. From the point where this incident ray strikes the mirror draw a line N at right angles to the surface of the mirror. This is the normal to the surface at the point of incidence.

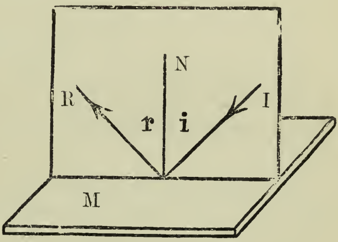


Fig. 148.

It is found that the incident ray, the normal and the reflected ray, are all in the same plane, in this special case the plane of the paper. If an incident ray moving along the plane of this paper fall upon a mirror, the normal to the mirror at the point of incidence and the reflected ray would not be found rushing out of the plane of the paper either down into the substance of the book or up towards the reader's eye. They would remain in the same plane as the incident ray.

Again, the angle *i* included between the incident ray I and the normal N is the angle of incidence. And the angle *r*, included between the reflected ray R and the normal N is the angle of reflection. Experiment has shown that these angles are equal in any given case.

Hence the two statements that simplify the study of the reflection of light run thus :—(1) The incident ray, the normal to the surface at the point of incidence, and the reflected ray are all in the same plane. (2) The angle of incidence = the angle of reflection.

C. Application to Mirrors.

We now proceed to apply these two great principles to the use of mirrors. A mirror is a reflecting surface, and may be either plane or curved.

(1) Plane Mirrors.

These are the mirrors best known to us, as the ordinary looking-glass is an illustration of plane mirrors. We will consider (*a*) one plane mirror; (*b*) combinations of plane mirrors.

(*a*) ONE PLANE MIRROR.

(i.) To find the position and nature of the image furnished by a plane mirror of an object. (ii.) To find the change produced in the position of a reflected ray by a change of position of the mirror.

(i.) *With one plane mirror to find the nature and position of the image.*—The image of an object is the collection of the foci of the reflected rays that reach the eye of the observer, and are due to incident rays from all points in the object. To make this definition more plain, imagine a man surveying himself in a pier-glass; from every point in his body rays of light are being thrown off, rays that came to his body primarily from the sun or other source of light. *Some* of these myriads of rays fall upon the surface of the looking-glass; these are reflected from the surface of the glass. *Some* of these reflected rays reach his eyes, and entering them, impinge upon the retina or sensitive membrane of the eye. The reflected rays due to certain incident rays from a given point of his body, if extended in their lines of direction, would all meet in a point (their focus). This point is not identical with any point in his body. It is an imaginary point behind the mirror. It will be the focus of the reflected rays, and will be the image of the special point we are considering in his body. The total collection of all these imaginary foci of the reflected rays that reach his eye and are the result of the reflection from the mirror of incident rays that set out from every point in his body, forms the image of his body.

Consider Fig. 149. OT represents an object in front of a plane mirror whose vertical section is represented by MR. From every point of the object, rays of light, infinite in number, are being given off in all directions. Multitudes fall upon the mirror. It will be sufficient for our purpose if we only consider the two extreme points of the object. If we can find the images of these, the image of the rest of the object will lie between these symmetrically with the object.

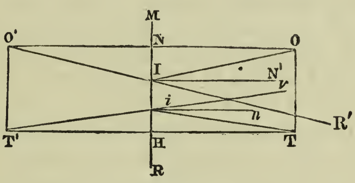


Fig. 149.

We seek therefore to find the image of the topmost point O of the object. Only two rays need be considered from this point, for the image of the point must be in the focus of the two reflected rays corresponding with these, or there will be no image at all. That is to say, the focus of *all* the reflected rays must be identical with that of these two reflected rays, or a confused blur will result. The same reasoning applies as to the number of rays necessary to be considered of the countless rays emanating from the lowest point.

Thus our problem simplifies down to considering only the extreme points of the object, and considering only two rays from each of those points. From O draw a line ON at right angles to the mirror. By the two principles given above, the incident ray represented by this line will be reflected upon itself. It makes with the normal an incident angle of 180°. The angle of reflection must be 180° also, *i.e.* the reflected ray must also be in the line of the normal, as is the incident ray. Somewhere in the line ON, or in its production in both directions, must be the image of the topmost point O. Next from the point O draw a line representing any other incident ray OI. Where it strikes the mirror, draw a normal IN′ to the mirror. The angle OIN′ is the angle of incidence of this ray. Make with the normal an angle N′IR′ = the angle of incidence OIN′. This new angle N′IR′ is the angle of reflection, and IR′ is the line of the reflected ray corresponding with the incident ray OI. Somewhere in the line IR′ or in its production in both directions must be the image of the point O; but as this image is in both the lines ON extended and IR′ extended, it can only be in the point of intersection of those two lines, *i.e.* at O′.

To find the image of the point T. From T draw a line TH at right angles to the mirror. By the two principles given above, the incident ray represented by this line will be reflected upon itself. It makes with the normal an incident angle of 180°. The angle of reflection must be 180° also, *i.e.* the reflected ray must also be in the line of the normal as is the incident ray. Somewhere in the line TH or in its production in both directions must be the image of the lowest point T. Next, from the point T draw a line T*i* representing any other incident ray. Where it strikes the mirror draw a normal *in* to the mirror. The angle T*in* is the angle of incidence of this ray. Make with the normal *in* an angle *nir* = the angle of incidence T*in*. This new angle *nir* is the angle of reflection, and *ir* is the line of the reflected ray corresponding with the incident ray T*i*. Somewhere in the line *ir* or in its production in both directions must be the image of the point T. But as this image is in both the lines TH extended and *ir* extended, it can only be in the point of intersection of those two lines, *i.e.* at T′.

Hence O′T′ will represent the image of the object OT. About this image and about all images, three questions must be asked.

1st. Is the image erect or inverted? Erect.

2d. Is the image magnified or diminished? Neither; it is of the same size as the object.

3d. Is the image real or virtual? Virtual; for the light does not really pass through it. Every one knows that the light does not really go behind the mirror, and a real image is one through which the rays of light do actually pass. A virtual image is one through which the rays of light only appear to pass.

To sum up, our image of an object in front of a single plane mirror is erect, of the same size as the object, virtual.

In making diagrams representative of the formation of images by reflection from mirrors, the student will do well to distinguish in some manner lines that represent actual rays of light from those that represent imaginary rays, or those that represent normals.

(ii.) *To find the change produced in the position of a reflected ray by a change of position of the mirror.*

Let MR represent a mirror in section that is capable of rotation round the point O in a plane identical with that of the paper. Let IO represent an incident ray normal to MR. This ray will be reflected upon itself along the line OI. If the mirror is turned through the angle α into the position M′R′, and IO still represents the incident ray, and ON the new normal to M′R′ at the point whereon IO impinges, $\angle$ ION = $\angle$ of incidence. But $\angle$ ION is the angle between two lines IO, ON perpendicular respectively to the mirror in its position MR and in its position M′R′, and therefore equals α. Again, the angle of reflection in this second case ($\angle$ NOR″) = $\angle$ ION = α;

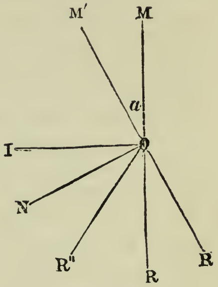


Fig. 150.

$$\therefore \text{ as } \angle \text{ION} = \alpha,$$
$$\text{and } \angle \text{NOR}'' = \alpha,$$
$$\angle \text{IOR}'' = 2\alpha.$$

But $\angle$ IOR″ is the angle between the two reflected rays OI and OR″, and this angle is equal to twice the angle through which the mirror has been rotated.

(*b*) Combinations of Plane Mirrors.

(i.) *When the mirrors have planes parallel one to another.*—If two plane mirrors be placed so that their planes are parallel, M, M′, Fig. 151, and an object OT be placed between them, an image 1 will be formed of OT by the mirror M at the same distance behind M as the object OT is in front of M. This image 1 will act as an object to the other mirror M′, and an image 1′ of 1 will be formed as far behind M′ as 1 appears to be in front of M′. This image 1′ will act as an object to the first mirror M, and an image 1″ of 1′ will be formed as far behind M as 1′ appears to be in front of M. This process will be repeated. And

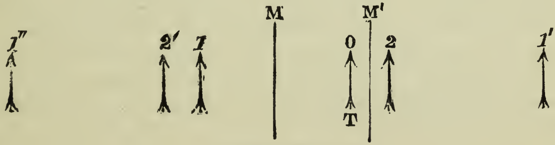


Fig. 151.

in like manner an image 2 will be formed of OT by the mirror M′ at the same distance behind M′ as the object OT is in front of M′. This image 2 will act as an object to the other mirror M, and an image 2′ of 2 will be formed as far behind M as 2 appears to be in front of M. This process will be repeated, and an almost endless succession of images of OT will appear.

(ii.) *When the mirrors are at an angle one with the other.*

(α) Let the angle = 90°. If MO′, M′O′ represent the mirrors, and O an object between them, an image 1 will be formed as far behind MO′ as O is in front of it; an image 2 will be formed as far behind M′O′ as O is in front of it. Each of these images 1 and 2 will form an image as far behind MO′ and M′O′ respectively, as 1 and 2 are in front of them, and these two new images will coincide in 3.

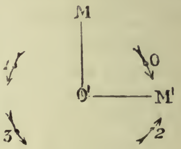


Fig. 152.

(β) Fig. 153 shows the formation of the images when the angle between the mirrors is 60°.

With 90° as angle of inclination of the mirrors, 3 images were formed,

$$360° \div 90° = 4.$$

With 60° as angle of inclination of the mirrors, 5 images are formed,

$$360° \div 60° = 6.$$

With 45° as angle of inclination of the mirrors, 7 images are formed. This the student should verify by diagram.

$$360° \div 45° = 8.$$

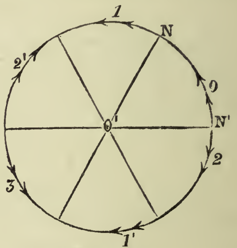


Fig. 153.

(γ) And generally, if the angle of inclination of the mirrors be some aliquot part of 360°, the number of images formed is one less than the quotient of 360° ÷ the number of degrees in the angle of inclination; *e.g.*, if the mirrors are arranged at an angle of 72° one with the other, as $360 \div 72 = 5$, $5 - 1$ or 4 images are formed.

(2) Spherical Mirrors.

In dealing with the subject of the reflection of light from the surface of spherical mirrors, we shall treat—(*a*) of the meaning of the normal in these mirrors; (*b*) of the principal axis and parallel pencils; (*c*) of concave mirrors; (*d*) of convex mirrors; (*e*) of the formulæ for mirrors.

(*a*) THE NORMAL IN SPHERICAL MIRRORS.

The normal has been defined as the perpendicular to the reflecting surface, drawn from the point of incidence of the incident ray. What then will be the normal in dealing with a spherical reflecting surface? The whole surface is made up of infinitely small and infinitely numerous particles. Each of these is common to the circle and to the tangent to the circle at the given point. Now the radius is perpendicular or normal to the tangent. The radius will be therefore considered as the normal to the reflecting spherical surface.

(*b*) PRINCIPAL AXIS AND PARALLEL PENCILS.

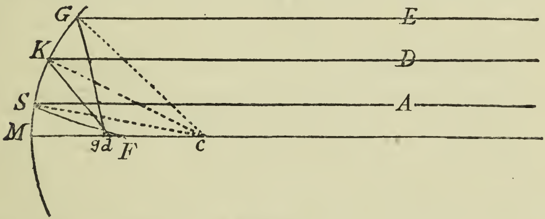


Fig. 154.

If the curve in Fig. 154 represent a vertical section of a concave mirror, and the curve in Fig. 155 represent a vertical section of a convex mirror, whereof C in each figure is the centre, the horizontal line passing through C and through the mirror is known as the principal axis of the mirror. The *principal axis* will always mean, therefore, the horizontal line passing through the centre of curvature of the mirror, and also through the mirror. Clearly, any ray of light traversing the line of this principal axis is in the line of the radius, *i.e.* in the line of the normal, and will be reflected upon itself. And indeed, more generally, any ray passing through the centre of curvature of the mirror, be the ray horizontal or oblique, is in the line of the radius or the normal, and must be reflected on itself.

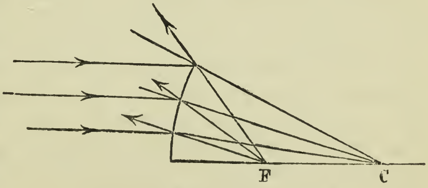


Fig. 155.

This is the first great guide in finding the position and nature of images with spherical mirrors.

Consider further the two Figs. 154, 155. A parallel pencil of incident rays is represented in each EG, DK, AS in the case of the concave mirror. All these are further parallel to the principal axis. From C normals are drawn to the various points of incidence of these rays. The angles EGC, DKC, ASC in Fig. 154 represent the angles of incidence. The respective angles of reflection CGF, CKF, CSF are drawn equal to the corresponding angles of incidence, and it is found that all the reflected rays pass through a point F on the principal axis. No matter how many incident rays are taken parallel to the principal axis, the corresponding reflected rays all go through the same point on the principal axis. This exceedingly important point, the focus of the reflected rays corresponding with a parallel pencil of incident rays, is called the *principal focus*, and is experimentally found to be midway between the centre of curvature C and the point in the mirror where the principal axis touches the mirror.

Looking at Fig. 155 where the convex mirror is represented, the normals drawn from C (on the opposite side to the incident rays) must be extended through the line representing the vertical section of the mirror. Then, when the angles of reflection are made equal to those of incidence, the reflected rays are seen to diverge from each other. They can never have a real focus on the same side of the mirror as is the luminous object whence proceed the incident rays. Their focus must be virtual, and is found by extending the lines of direction of the reflected rays till they meet at F, a point on the principal axis. No matter how many incident rays are taken parallel to the principal axis, the corresponding reflected rays if extended beyond the mirror all go through the same point on the principal axis. This exceedingly important point, the focus of the reflected rays corresponding with a parallel pencil of incident rays, is called the *principal focus*, and is experimentally found to be midway between the centre of curvature C and the point in the mirror where the principal axis touches the mirror.

The distance, measured along the principal axis, of the principal focus from the mirror is called the *focal length*, and is equal to one-half the radius. This fact is the second great guide in finding the position and nature of images with spherical mirrors.

To sum up:—The normal is the radius of curvature drawn to the point of incidence.

The principal axis is the horizontal line drawn through the centre to the mirror.

The principal focus is the focus of reflected rays due to a parallel incident pencil.

The focal length is the distance from the principal focus to mirror.

The focal length = one-half the radius.

(*c*) CONCAVE MIRRORS.

To find the position and nature of the image.—Three cases present themselves—(i.) where the object is beyond the centre of curvature; (ii.) where the object is between the centre of curvature and the principal focus; (iii.) where the object is between the principal focus and the mirror.

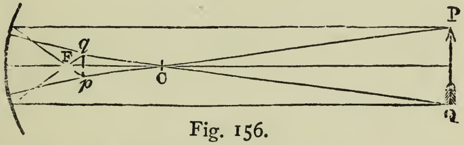


Fig. 156.

(1.) Let the curved line in Fig. 156 represent a vertical section of a concave mirror whose centre of curvature is at C and principal focus at F. Let PQ represent the object, beyond C. It is required to find the position and nature of the image of PQ formed by the mirror, that is, the position of the foci of the reflected rays that reach the eye of the observer, and are due to certain incident rays emanating from PQ.

As with plane mirrors, only the extreme points P and Q need be considered, and only two incident rays from each point. From P draw a line representing a ray that passes through C and impinges on the mirror. This ray PC will be reflected on itself. Somewhere in the line of this ray will be the image of P. Next, from P draw a horizontal ray. This, after striking the mirror, will be so reflected as to pass through F. Somewhere in the line of this reflected ray will be the image of P. As the image therefore of P is somewhere in two different lines, it can only be where these intersect, *i.e.* at *p*.

In like manner from Q draw a line representing a ray that passes through C and impinges on the mirror. This ray QC will be reflected on itself. Somewhere in the line of this ray will be the image of Q. Next, from Q draw a horizontal ray. This, after striking the mirror, will be so reflected as to pass through F.

Somewhere in the line of this reflected ray will be the image of Q. As the image therefore of Q is somewhere in two different lines, it can only be where these intersect at q.

pq will represent the image of PQ. The answers to the three questions, Is the image erect or inverted, magnified or diminished, real or virtual? are: It is inverted, diminished, real.

(ii.) *Where the object is between the centre of curvature and the principal focus.*

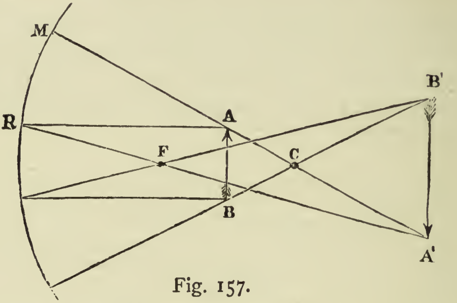


Fig. 157.

From A draw a ray AM to the mirror in the direction of the radius CAM. The ray represented by this line is reflected upon itself. The image of A must be somewhere in the line MC produced. From A also draw AR parallel to the principal axis. This ray is reflected through F. The image of A must be in the line RF produced. Hence, as this image is in the lines MC and RF, it must be at the point A′ where these lines intersect.

Proceeding in like manner with rays from the point B, the image of B is found to be at B′, and A′B′ represents the image of AB. This image is inverted, magnified, and real.

(iii.) *Where the object is between the principal focus and the mirror.*—From O draw OR to the mirror in the direction of the

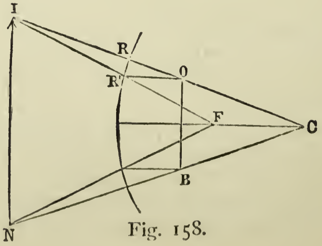


Fig. 158.

radius COR. The ray represented by this line is reflected upon itself, and the image of the point O must be somewhere in the line COR produced. From O draw OR′ parallel to the principal axis. The ray thus represented will be reflected through F, the principal focus and the image of the point O will be somewhere in the line FR′ or its production. Hence the image of O must be at I beyond the mirror where these two lines COR, FR′ intersect.

Proceeding in like manner with rays from B, the image of B is found to be at M, and IM represents the image of OB. This image is erect, magnified, virtual.

(*d*) CONVEX MIRRORS.

Though the positions of the object in relation to a convex mirror fall under three heads, the nature of the image will be found to be the same, except as to size, in all three cases. Case (i.), where the object is at a distance from the mirror greater than the radius of the mirror, will be fully explained; cases (ii.) and (iii.) will be illustrated by diagrams only.

(i.) *When the object is at a distance from the mirror greater than the radius.*—Let the curved line in Fig. 159 represent a vertical section of a convex mirror whose centre of curvature is at C, and principal focus at F. C′ and F′ are the corresponding points on the side of the mirror on which the object is. Let AB represent the object, beyond C′.

From A draw a line representing a ray that if extended would pass through C. This ray is reflected on itself, and somewhere in the line AC or its production will be the image of A. Next, from A draw a horizontal ray. This after striking the mirror will be so reflected that the extension of the reflected ray will pass through F.

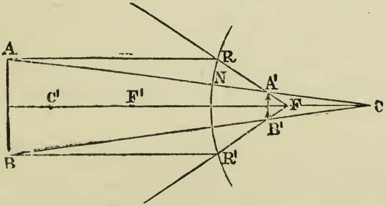


Fig. 159.

Somewhere in the line FR, or its production, will be the image of A. Hence this image must be at A′, where the lines CA, FR intersect.

Proceeding in like manner with two rays BC and BR′ from B, its image must be at B′.

This image is erect, diminished, virtual.

(ii.) *When the object is at a distance from the mirror less than the radius, but greater than the focal length.*

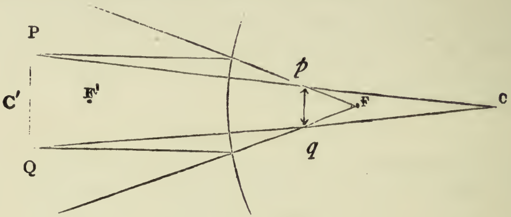


Fig. 160.

(iii.) *When the object is at a distance from the mirror less than the focal length.*

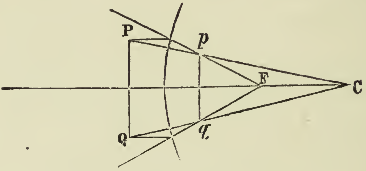


Fig. 161.

(*e*) Formulæ for Mirrors.

(i.) Especially in connection with the three cases of convex mirrors, it will have been noticed that the magnitude of the image varies under different conditions, and the student will have observed that there is some sort of relationship between the sizes of object and of image and the positions of object and image. It is necessary now to state this relationship in more definite terms.

In Fig. 162, PQ represents the linear magnitude of the object,

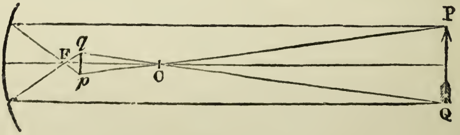


Fig. 162.

pq the linear magnitude of the image.

In the two triangles CPQ, C*pq*,

Angles CQP, C*qp* are equal as alternate angles,
Angles CPQ, C*pq* ,, ,,
Angles PCQ, *p*C*q* ,, vertically opposite;

∴ triangles CPQ, C*pq* are equiangular and are similar;
∴ their sides and their heights are proportional;

$$\therefore \frac{PQ}{pq} = \frac{\text{height of } \triangle\ CPQ}{\text{height of } \triangle\ Cpq} = \frac{\text{distance of object from C}}{\text{distance of image from C}},$$

and the linear magnitudes of object and image are as the distances of object and image respectively from the centre of curvature or from the mirror.

Similar reasoning applies to convex mirrors. Turning to Fig. 163, the above reasoning applies word for word and letter for letter, save that the equality of angles CPQ, C*pq* and of angles CQP, C*qp* depends upon the members of the respective pairs being internal and external and opposite angles respectively, and angles PCQ, *p*C*q* are the same.

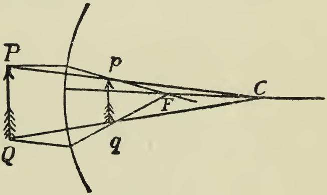


Fig. 163.

Generally, therefore, the linear magnitudes of object and image in the case of mirrors are proportional to their distances from the centre of curvature or from the mirror.

(ii.)

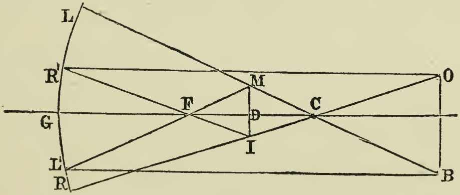


Fig. 164.

Consider △ BML′. FC is parallel to the base BL′; ∴ it cuts the sides BM, ML′ proportionally.

$$\therefore \frac{MC}{CB} = \frac{MF}{FL'}.$$

Consider △ MCI and △ OCB. They are equiangular.

$$\therefore \frac{MC}{CB}=\frac{MI}{OB}=\frac{\text{size of image}}{\text{size of object}}.$$

If the portion of the mirror used be very small, MF approximately = DF = distance of image from principal focus, and FL′ approximately = FG = focal length.

$$\text{Hence } \frac{MF}{FL'}=\frac{\text{distance of image from principal focus}}{\text{focal length}}, \text{ and } \frac{MC}{CB}$$

$$=\frac{MF}{FL'} \text{ becomes } \frac{\text{size of image}}{\text{size of object}}$$

$$=\frac{\text{distance of image from principal focus}}{\text{focal length}}.$$

In like manner it can be shown that the size of the object is to the size of the image as the distance of the object from the principal focus is to the focal length.

(iii.) It is evident that much trouble in the calculation of the relative magnitudes of object and image will be saved, if, without the trouble of making diagrams and actual measurements every time, some simple formula can be devised, connecting together the distances of object and of image from the mirror and the focal length. The student will do well to make at first diagrams in every case, but after a time the use of a formula (when the principles are fully understood) saves much time.

Consider Fig. 165. The curved line represents a vertical section of a concave mirror, C its centre of curvature, F the principal focus, O a luminous point in the object, I the point in the image corresponding to O, CM a line through C parallel to OR.

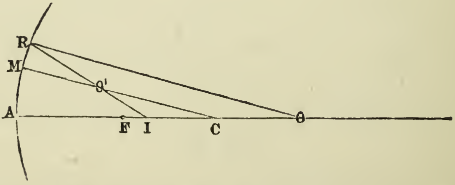


Fig. 165.

Let us agree to call the radius of the circle r, the distance of O from the mirror u, the distance of I from the mirror v, the focal length f.

Hence CA $= r$, OA $= u$, IA $= v$, FA or CF $= f$.

In the triangles CO′I, ORI,

Angle CIO′ = angle OIR (identical angles),

Angle CO′I = angle ORI (external = internal and opposite angle),

Angle ICO′ = angle IOR (" ");

∴ triangles CO′I, ORI are equiangular and similar;

∴ their sides are proportional;

$$\therefore \frac{\text{CI}}{\text{CO}'} = \frac{\text{OI}}{\text{OR}}.$$

But CI = CA − IA $= r - v$.

CO′ (if the part of the mirror considered be small) = CF $= \frac{r}{2}$ (approximately).

OI = OA − IA $= u - v$.

OR (if the part of the mirror considered be small) = OA $= u$ (approximately);

$$\therefore \frac{\text{CI}}{\text{CO}'} = \frac{\text{OI}}{\text{OR}} \text{ becomes } \frac{r-v}{\frac{r}{2}} = \frac{u-v}{u}.$$

But r = twice focal length $= 2f$, and $\frac{r}{2} = f$;

$$\therefore \frac{2f-v}{f} = \frac{u-v}{u};$$

∴ cross-multiplying, $2uf - uv = uf - vf$;

∴ transposing, $uf + vf = uv$;

∴ dividing by uvf, $\frac{uf}{uvf} + \frac{vf}{uvf} = \frac{uv}{uvf}$;

$$\therefore \frac{1}{v} + \frac{1}{u} = \frac{1}{f},$$

i.e. the reciprocal of the distance of the image from the mirror + the reciprocal of the distance of the object from the mirror = the reciprocal of the focal length.

As $f = \frac{r}{2}$, $\frac{1}{f}$ may be written $\frac{1}{\frac{r}{2}}$ or $\frac{2}{r}$.

Sometimes, therefore, the formula runs

$$\frac{1}{v}+\frac{1}{u}=\frac{2}{r}.$$

These formulæ are true of both concave and convex mirrors, only the student must remember that all focal lengths of convex mirrors must be regarded as negative quantities, as they are measured on the side of the mirror opposed to that whereon the object is.

Example 122.—A concave mirror has a focal length of 8 inches. An object is placed 2 inches from the mirror. Determine the position of the image.

The formula required is $\frac{1}{v}+\frac{1}{u}=\frac{1}{f}$.

v is unknown, $u=2$ inches, $f=8$ inches.

$$\frac{1}{v}+\frac{1}{2}=\frac{1}{8},$$

$$\frac{1}{v}=\frac{1}{8}-\frac{4}{8},$$

$$\frac{1}{v}=-\frac{3}{8};$$

$$\therefore\ v=-\frac{8}{3}=-2\frac{2}{3}\text{ inches};$$

∴ the image is on the side of the mirror opposed to that on the which is the object, and $2\frac{2}{3}$ inches from the mirror.

Example 123.—A concave mirror whose radius of curvature is 1 foot, has an object placed before it, and the image of that object is thrown upon a screen 6 inches farther from the mirror than the centre. Determine the position of the object.

The formula required is $\frac{1}{v}+\frac{1}{u}=\frac{2}{r}$.

$r=12$ inches, $v=18$ inches, u is unknown.

$$\frac{1}{v}+\frac{1}{u}=\frac{2}{r},$$

$$\frac{1}{18}+\frac{1}{u}=\frac{2}{12}=\frac{1}{6},$$

$$\frac{1}{u}=\frac{1}{6}-\frac{1}{18},$$

$$\frac{1}{u}=\frac{3-1}{18},$$

$$\frac{1}{u}=\frac{2}{18}=\frac{1}{9};$$

$$\therefore\ u=9 \text{ inches.}$$

Example 124.—A convex mirror forms an image of an object when that object is 1 foot from the mirror. If the image is $\frac{1}{4}$ the magnitude of the object, what is the focal length of the mirror?

Size of image : size of object $::$ distance of image from mirror : distance of object from mirror;

$$\therefore\ \tfrac{1}{4} : 1 :: x : 1 \text{ foot},$$

$$\tfrac{1}{4}\text{ foot}=x,$$

$$x=3 \text{ inches.}$$

$$\frac{1}{v}+\frac{1}{u}=-\frac{1}{f}\ (\text{mirror is convex}),$$

$$\frac{1}{3}+\frac{1}{12}=-\frac{1}{f},$$

$$\frac{4+1}{12}=-\frac{1}{f};$$

$$\therefore\ f=-\frac{12}{5} \text{ or } -2\frac{2}{5}.$$

EXERCISES.

1. Find radius of a concave mirror that forms an image at 5 dcm. distance, when the object is at 12 dcm. distance on same side as image.

2. A concave mirror of diameter 20 inches has an object 3 inches in front of it. Determine the position of the image.

3. Where must an object be placed in respect to a convex mirror of radius 12 cm. that the image may appear to be distant from the mirror 2 cm., and on the opposite side to the object?

4. If with a concave mirror the size of the image is to that of the object as 6 : 5, and the object is distant 4 cm. from the mirror, determine the position of the image and the diameter of the mirror.

CHAPTER XXVII.

V. REFRACTION.

As long as light is passing through the same medium, whether that medium be air, water, or glass, so long the light passes in straight lines. But when light, after passing through a certain medium, enters another differing from the first in density, the light is bent out of its course at the moment of entry. After the light has once passed into the new medium, it will continue to move in a straight line therein; but at the moment of passing into or of passing out from any given medium, the light is altered in direction. This change of direction, as light changes its medium, is called refraction.

We shall consider—A. The laws of refraction; B. Certain phenomena resulting thence; C. Lenses; D. Formulæ in respect to lenses.

A. THE LAWS OF REFRACTION.

It is noticed that whenever light passes from a rarer medium to a denser one, it is bent *towards* a line normal to the surfaces of the two media at the point of incidence of the ray of light. Thus in Fig. 166, if *i* represent an incident ray of light that after passing through air impinges upon the surface of water, on passing into the denser water that ray is refracted towards the normal *n* drawn perpendicular to the line of junction of the surfaces of the air and water at the point of incidence of *i*. It takes the direction of *r*. On the other hand, whenever light passes from a denser medium to a rarer one, it is bent *away from* the

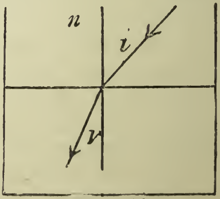


Fig. 166.

normal. Thus, if the light were passing from the water below into the air above in Fig. 166, *r* would represent the incident ray and *i* the refracted. *i* evidently is farther away from *n* the normal than is *r*.

Generally, therefore, light passing from rarer to denser medium is refracted towards normal, and light passing from denser to rarer medium is refracted from normal.

To obtain statements as to refraction of more definite and numerical value than this general one, experiments have been made. In Fig. 167 the horizontal line represents the surface of contact of two media, whereof the upper is the rarer, the lower the denser. AO is an incident ray; *mn*, the normal to the line of junction of the two surfaces at the point of incidence of the ray AO; OB, the refracted ray, bent towards *mn*. First, it is found that AO, *mn*, OB are all in the same plane. Hence the first 'law' of refraction is that (1) *the incident ray, the normal, and the refracted ray are all in the same plane.*

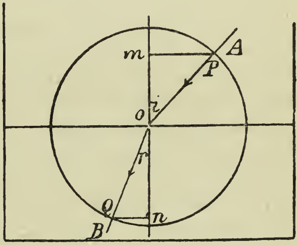


Fig. 167.

Let us now attempt to find some relationship between the angles of incidence and refraction *i* and *r*. They are evidently not equal, as are the angles of incidence and reflection in the case of reflection of light. With O as centre, describe any circle cutting the lines of direction of the incident and refracted rays and that of the normal. From P and Q, where this circle cuts the incident and refracted rays respectively, draw horizontal lines P*m*, Q*n* perpendicular to the normal *mn*. In trigonometrical phrase,

$$\frac{Pm}{PO} = \text{the sine of the angle } i,$$

$$\frac{Qn}{QO} = \quad " \quad " \quad r.$$

Now it is found that, given the same two media, no matter what is the direction of the incident ray, the ratio of the sine of the angle of incidence to the sine of the angle of refraction never varies; it is a constant. The angles vary, their sines vary, but the *ratio* of the sines in all cases one to another never varies.

Looking at $\frac{Pm}{PO}$, the sine of angle i,

and $\frac{Qn}{QO}$, ,, ,, r,

it will be observed that the denominators PO, QO are equal, as they are both radii of the same circle;

∴ the *ratio* of the sines of the angles i and $r = \frac{Pm}{Qn}$.

Pm and Qn are not the sines of the angles of incidence and refraction, but they are proportional to them. For this long periphrasis, 'the ratio of the sine of the angle of incidence to the sine of the angle of refraction for any two given media,' the term 'refractive index' is used. Hence the second 'law' of refraction runs that (2) *for any two given media the refractive index is a constant.*

To sum up, the laws of refraction are—

(1) The incident ray, the normal, the refracted ray, are all in the same plane.

(2) The refractive index for any two given media is a constant.

The refractive index for rays passing from air to glass $= \frac{3}{2}$,
,, ,, ,, ,, glass to air $= \frac{2}{3}$,
,, ,, ,, ,, air to water $= \frac{4}{3}$,
,, ,, ,, ,, water to air $= \frac{3}{4}$.

Example 125.—Given that the refractive index of two media is $\frac{4}{5}$, make a diagram representing the passage of a ray of light from the rarer to the denser.

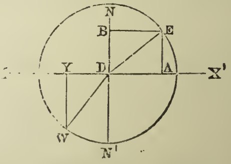


Fig. 168.

Draw a horizontal line XX′, draw a normal to it, NN′, intersecting XX′ in D. Let the ray be ED. Along DX′ measure off 5 units = DA; draw AE perpendicular to DA, meeting DE in E. Complete the parallelogram DAEB; on DX measure off DY, containing 4 of the units selected. With centre D at distance DE, describe a circle; from Y drop a perpendicular YW, meeting the circumference of the circle in W. Join DW; then DW is the course of the refracted ray, bent towards the normal NN′.

B. CERTAIN PHENOMENA RESULTING FROM THE LAWS OF REFRACTION.

These will comprise—(1) the apparent change of depth of a vessel when filled with a liquid; (2) the distortion of objects partially immersed in a liquid; (3) total reflection.

1. The Apparent Change of Depth of a Vessel when a Liquid is poured into it.

A basin more or less completely filled with water appears to be shallower than it is in reality. A tank at an aquarium seems to be of less dimensions from front to back than it is in reality.

In Fig. 169 let WAET represent a vessel containing water, and B a point in its base. From B myriads of rays are passing. Consider two of them; one, BV, passing vertically out of the water into the air and not refracted at all as it is in the line of the normal; another, BO, which on quitting the water is bent from the normal in the direction OR. The image of B must be in the two lines BV and OR, or in their productions. It can only be, therefore, at B′, the point of intersection of the twain. Hence B appears to be at B′. Similarly with every point in the base of the vessel of water. Hence the vessel appears to be more shallow than it is in reality.

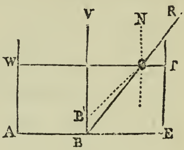


Fig. 169.

2. The Distortion of Objects partially immersed in a Liquid.

The stock example of this is the stick plunged partly into water. In Fig. 170, SK represents a stick partly in air, partly in water. The part SO′ in air is seen in the ordinary manner; let us study the part O′K. Consider K. By reasoning so similar to that under the last head as not to need repetition, K will appear to be at K′. Similarly with every point between K and O′. Hence the stick appears as if it were bent at O′, and SO′K′ were its line.

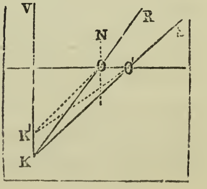


Fig. 170.

3. Total Reflection.

This phenomenon occurs in connection with rays passing from a denser to a rarer medium. Consider Fig. 171. The light is passing from a point in a denser medium out into a rarer one. The angle *r* is less than angle *i*. Suppose the incident ray under consideration to shift up to the left. As its incident angle *r* increases, the refracted angle *i* increases, and the refracted ray OA moves farther and farther downwards to the right, and approximates more and more closely to the horizontal line of juncture of the two media. At length, if the incident ray BO be shifted sufficiently to the left, the incident angle *r* becomes of such size that the refraction angle *i* is a right angle, and the refracted ray OA is horizontal, and glints along the surface of juncture of the two media. When that occurs, the incident angle is called the *critical angle.*

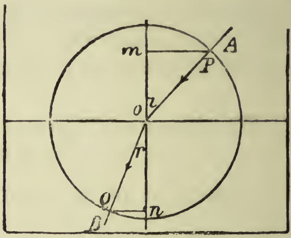


Fig. 171.

The critical angle is therefore the angle of incidence of a ray of light passing from a denser medium into a rarer when the angle of refraction = 90°.

But what is to happen when the angle of incidence under these circumstances is greater than the critical angle, and the incident ray BO has shifted yet farther up to the left? Refraction cannot occur. Experiment shows that reflection takes place from the surface of contact of the two media back into the substance of the denser medium. That is to say, the incident ray BO does not pass out into the rarer medium, but is reflected from the contact surface back into the denser medium, according to the ordinary laws of reflection of light. As all the light is reflected, and none passes into the rarer medium and undergoes refraction, this is called total reflection.

Total reflection, therefore, is the reflection of light from the contact surface of two given media when the incident ray is in the denser medium and the angle of incidence is greater than the critical angle for the two given media.

C. LENSES.

In studying lenses we will—(1) define a lens, (2) describe the various kinds of lenses known, (3) consider the formation of images by lenses.

(1) Definition of a Lens.

A lens is a portion of a refracting medium with two surfaces, one of which at least must be curved. Both may be curved, or one may be curved and one plane. Lenses are used for the purpose of collecting or of dispersing rays of light. They are generally made of glass, though within the eye of man are certain lenses not made of glass.

(2) Kinds of Lenses.

(*a*) There are two genera of lenses, each containing three species. The three lenses represented to the left in Fig. 172 are all most thick in the middle, all cause rays of light to converge, all are called convex lenses. (i.) That represented most to the left has one surface plane, the other curved; it is called a plano-convex lens. (ii.) That represented next to it has both surfaces curved in opposite directions; it is called a doubly-convex lens. (iii.) The convex lens most to the right has both surfaces curved in the same direction; it is called a concavo-convex lens, or converging meniscus.

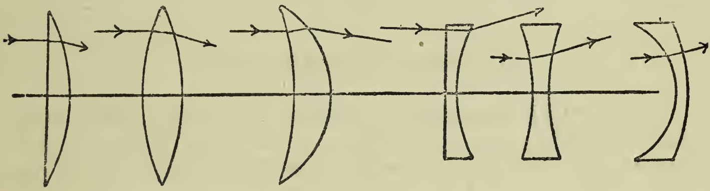


Fig. 172.

(*b*) The three lenses represented to the right are all called concave lenses. (i.) The first of these, counting from the left, has one surface plane, the other curved; it is called a plano-concave lens. (ii.) That represented next to it has both surfaces curved in opposite directions; it is called a doubly-concave lens. (iii.) The lens most to the right of the whole six has both surfaces curved in the same direction; it is called a convexo-concave lens or diverging meniscus.

It will be observed that the three convex lenses are thickest in the middle, and they all cause the rays of light to converge, whilst the concave lenses are thinnest in the middle, and all cause the rays of light to diverge.

(3) The Formation of Images by Lenses.

In this connection we shall only study one lens of each genus, viz. the one with the two surfaces curved in opposite directions, and for simplicity's sake shall assume that the lenses have no thickness. To the actual centre of a lens, not the centre of curvature, but the centre of structure, the name *optical centre* is given. Rays passing through this optical centre may be regarded, if the lens be supposed to have no thickness, as not undergoing any refraction at all, but as passing straight through the lens.

(*a*) DOUBLE CONVEX LENSES.

In Fig. 173 is represented a double convex lens with optical centre O and centres of curvature of its two surfaces C and C′ respectively. Consider a parallel pencil of incident rays falling upon the lens from the right-hand side. The horizontal ray passing through C and O is not refracted at all. This undeviating path of this particular ray is due not to its passing through C or C′, but to its passing through O the optical centre. The line

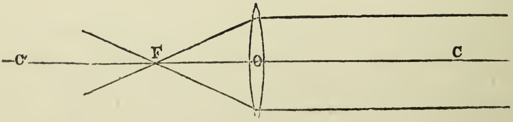


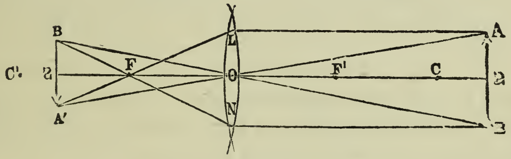
Fig. 173.

COC′ joining the optical centre and the centres of curvature is the principal axis. All the other rays parallel to COC′ are made to converge, and it is found experimentally that if the lens be not too large, the refracted rays all cut the ray COC′ in one point F. This point is therefore the focus of the refracted rays resulting from the action of a double convex lens upon a parallel pencil, and is called the principal focus. The point F is experimentally found to be midway between the lens O and its centre of curvature C′, so that the focal length FO in the lens as in the mirror is one-half the radius.

We have now two rays that will serve as guides in determining the nature and position of an image due to the action of a double convex lens on the rays of light proceeding from any object—viz. a ray passing through the optical centre and not refracted at all, and a ray parallel to the principal axis that is refracted to the principal focus.

In studying the formation of images by a double convex lens, three cases present themselves—(i.) when the object is beyond the centre of curvature; (ii.) when the object is between the centre of curvature and the principal focus; (iii.) when the object is between the principal focus and the lens.

(i.) *Object beyond centre of curvature.*



F g. 174.

Let AB represent an object placed beyond the centre of curvature C of a double convex lens whose optical centre is at O. Let F represent the principal focus, C′ the centre of curvature on the left-hand side of the lens, and F′ the principal focus if the pencil of parallel rays were proceeding from left to right.

The image of AB will be the collection of the foci of all the refracted rays that reach the eye of the observer, and are due to the action of the lens on rays emanating from every point in AB. Only the two extreme points need be considered, and only two rays from each of those points, for reasons given on page 211.

From A draw a line representing a ray passing through the optical centre O; this ray will not undergo any refraction. Therefore the image of A will be somewhere in the line AO, or its production. From A draw a line representing a horizontal ray parallel to the principal axis; this ray will be refracted to F. Somewhere in the line LF must be the image of A. As this image is in both the lines AO produced and LF produced, it can only be at A′ where they intersect.

Consider now the point B. The ray BO is not refracted as it passes through the optical centre. The ray BN is refracted to F. The image of B is therefore at B′, the point where these two lines BO produced and NF intersect. The image of AB is represented by A′B′.

In reply to the three usual questions, we can state that—(α) it is inverted; (β) it is diminished; (γ) it is real (the light does actually pass through it, for the eye of the observer is beyond C′, and grasps the rays of light after they have again diverged from A′B′). This case is met with in the astronomical telescope.

(ii.) *Object between centre of curvature and principal focus.*

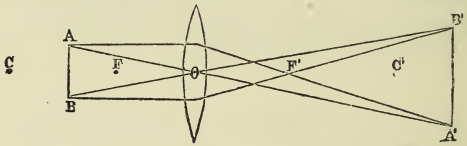


Fig. 175.

The image is inverted, magnified, real. The eye is supposed to be beyond A′B′ to the right, and to receive diverging rays that have actually passed through A′ and B′.

(iii.) *Object between principal focus and lens.*

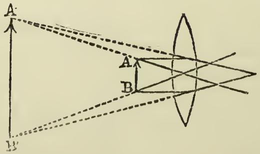


Fig. 176.

The rays from AB, after passing through the lens and undergoing refraction, would never meet on the side of the lens remote from the object. They only meet when produced on the side of the lens on which the object is; hence the image A′B′ is erect, magnified, virtual. This is the case of the ordinary magnifying-glass.

(*b*) DOUBLE CONCAVE LENSES.

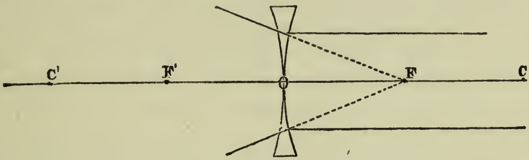


Fig. 177.

In Fig. 177 is represented a double concave lens with optical centre O, and centres of curvature of its two surfaces C and C′ respectively. Consider a parallel pencil of incident rays falling upon the lens from the right-hand side. The horizontal ray passing through C and O is not refracted at all, because it passes through O, the optical centre. The line COC′, joining the optical centre and the centres of curvature, is called the principal axis. All other rays parallel to COC′ are made to diverge, and it is found experimentally, if the lens be not too large, that the lines of the refracted rays which can never have a real focus all meet when produced at a point F on the principal axis. F is therefore the principal focus of the lens, and FO is the focal length. The focal length is one-half the radius.

Once more we have two rays that will aid in determining the nature and position of an image formed by a double concave lens—viz. one passing through the optical centre and not refracted at all, and a ray parallel to the principal axis that is refracted to the principal focus.

To find the nature and position of the image.

(i.) *When the object is beyond the centre of curvature.*

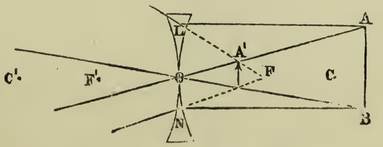


Fig. 178.

Let AB represent an object placed beyond the centre of curvature C of a double concave lens whose optical centre is at O. Let F represent the principal focus, and C′F′ the points on the opposite side of the lens corresponding with C and F.

Consider only the extreme points of the object and two rays from each. Ray AO is not refracted. Ray AL is refracted in such a direction that the extension of the refracted ray passes through F. The image of A will be at A′, the point of intersection of AO and LF. Again, ray BO is not refracted. Ray BN is refracted in such a direction that the extension of the refracted ray passes through F. The image of B will be at B′, the point of intersection of BO and NF.

The image is erect, diminished, virtual. The eye is to the left of the lens, and grasps the diverging rays.

(ii.) *When the object is between the centre of curvature and the principal focus.*

It will only be necessary to make the diagram in this and the next case. The student will do wisely in studying the diagrams, and reasoning out for himself the formation of the image. He must in no manner consider himself master of the subject until he is able to make each and all of these diagrams himself.

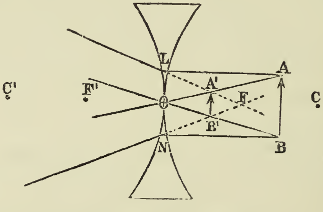


Fig. 179.

The image is erect, diminished, virtual.

(iii.) *Object between principal focus and lens.*

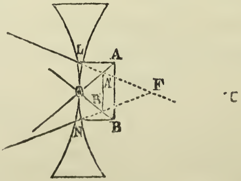


Fig. 180.

The image is erect, diminished, virtual.

It will be observed that the relative sizes of image and object vary in the six cases that have been studied. It remains to prove and state the exact relationships obtaining between the relative sizes of object and image and their positions.

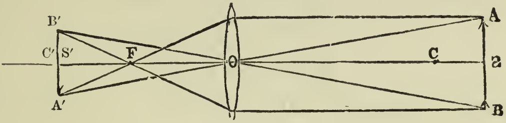


Fig. 181.

In Fig. 181 consider the triangles OAB, OA′B′.

$\angle AOB = \angle A'OB'$ (vertically opposite angles),
$\angle OAB = \angle OA'B'$ (alternate angles),
$\angle OBA = \angle OB'A'$ (alternate angles);

$\therefore$ triangles OAB, OA′B′ are similar, and their sides and heights proportional;

$\therefore \dfrac{AB}{A'B'} = \dfrac{OS}{OS'}$, *i.e.* size of object : size of image :: distance of object from optical centre of lens : distance of image from optical centre of lens.

A similar proof holds in respect to concave lenses, and generally the sizes of images and objects with lenses are as the distances of images and objects from the lens.

(*c*) FORMULÆ FOR LENSES.

It is evident that much trouble in the calculation of the relative magnitudes of object and image will be saved, if, without making diagrams and actual measurements every time, some simple formula can be devised, connecting together the distance of the object from the lens, the distance of the image from the lens, and the focal length of the lens.

Consider Fig. 182.

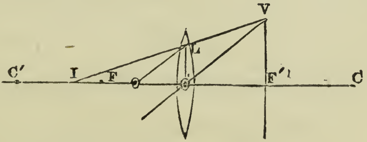


Fig. 182.

O′ is the optical centre of a double convex lens, C and C′ centres of curvature, F, F′ principal foci, CO′C′ the principal axis O an object point thereon, I the image of the point O.

Let us agree to call the distance of O from the lens u, the distance of I from the lens v, the focal length f.

$$\text{Hence } OO' = u,$$
$$IO' = v,$$
$$FO' \text{ or } F'O' = f.$$

Consider first, triangles OLO′, O′VF′.

$\angle LO'O = \angle VF'O'$ (right angles),

$\angle OLO' = \angle O'VF'$ (external and internal and opposite angles),

$\angle O'OL = \angle F'O'V$ (external and internal and opposite angles);

∴ triangles OLO′, O′VF′ are similar, and their sides proportional;

$$\therefore \frac{LO'}{O'O} = \frac{VF'}{F'O'}. \quad . \quad . \quad . \qquad \text{(i.)}$$

Next, consider triangles ILO′, IVF′.

$\angle LO'I = \angle VF'I$ (right angles),

$\angle ILO' = \angle IVF'$ (external and internal and opposite angles),

$\angle LIO' = \angle VIF'$ (identical angles);

∴ triangles ILO′, IVF′ are similar, and their sides proportional;

$$\therefore \frac{LO'}{O'I} = \frac{VF'}{F'I}. \quad . \quad . \quad . \quad . \qquad \text{(ii.)}$$

Divide equation (i.) $\frac{LO'}{O'O} = \frac{VF'}{F'O'}$ by (ii.) $\frac{LO'}{O'I} = \frac{VF'}{F'I}$,

$$\text{and } \frac{LO'}{O'O} \times \frac{O'I}{LO'} = \frac{VF'}{F'O'} \times \frac{F'I}{VF'};$$

$$\therefore \frac{O'I}{O'O} = \frac{F'I}{F'O'}.$$

$$\text{But } F'I = F'O' + O'I;$$

$$\therefore \frac{O'I}{O'O} = \frac{F'O' + O'I}{F'O'}. \quad . \quad . \qquad \text{(iii.)}$$

Substituting u, v, f, for the corresponding lines in (iii.),

$$\frac{v}{u} = \frac{f+v}{f}.$$

Cross multiplying, $vf = uf + uv$.

Dividing by uvf, $\frac{1}{u} = \frac{1}{v} + \frac{1}{f}$.

But in convex lenses f = always negative;

$$\therefore \frac{1}{u} = \frac{1}{v} - \frac{1}{f};$$

$$\therefore \frac{1}{u} - \frac{1}{v} = -\frac{1}{f}.$$

$$\text{Changing signs, } \frac{1}{v} - \frac{1}{u} = \frac{1}{f}$$

This formula $\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$ is general for lenses. Always must the student keep in mind that lines measured to the lens in the same direction as the light is travelling are negative, lines measured in the opposite direction to that in which the light is travelling are positive.

Example 126.—Find the position of the image formed by a double concave lens of radius one foot, if an object be placed 3 inches from the lens.

Given a double concave lens of radius 12 inches, ∴ focal length 6 inches; an object 3 inches from lens; it is required to find the position of the image.

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}; \ v \text{ is unknown, } u = 3, f = 6.$$

$$\frac{1}{v} - \frac{1}{3} = \frac{1}{6}; \therefore \frac{1}{v} = \frac{1}{6} + \frac{1}{3} = \frac{1}{2};$$

∴ $v = 2$, and the image is 2 inches from the lens.

EXERCISES.

1. Find the focal length of a double concave lens, forming at a distance of 1 decimetre from itself an image of an object distant 3 dcm.

2. With a double concave lens of diameter 2 metres, find the position of the image when the object is 4 dcm. from the lens.

3. With the same data as in Question 1, save that the lens is double convex and the object and image on opposite sides, calculate the focal length.

4. A double convex lens is used as a magnifying glass. Its focal length is 5 cm. If the object is within 2 cm. of the lens, calculate the relative sizes of image and object.

HEAT.

The last subject claiming our consideration is the set of phenomena known under the general name of Heat. We shall consider —I. The nature of heat and temperature. II. Instruments used in connection with the investigation of heat and of temperature. III. The effects of heat. These will comprise expansion, the change of state from solid to liquid, and from liquid to gas, under the continued action of heat, and specific heat. IV. The diffusion of heat by conduction, convection, and radiation.

CHAPTER XXVIII.

I. THE NATURE OF HEAT AND TEMPERATURE.

In the former times heat was said to be due to a form of matter, a fluid called Caloric. To-day heat is regarded as due to motion, motion of the molecules of bodies. Heat, like electricity and other allied forces, is a result of molecular vibrations.

Temperature must not be confounded with amount or quantity of heat. A familiar example will show they are not identical. If a thermometer be immersed in a large vessel of boiling water, and in a cup of boiling water, it registers the same *temperature*, although it is evident that the amounts of *heat* in the large vessel and in the cup of boiling water are very different. Temperature is rather a state or condition of a body. It has been defined as the condition of a body in respect to sensible heat, for heat is often present in substances, though it is not evident to the senses. The temperature of any solid, liquid, or gas will therefore always mean to us a certain condition of that solid, liquid, or gas in respect to heat that can be perceived by the senses; and we pass at once to the study of certain instruments whose office is to record that condition. It will then be possible for us to obtain clear ideas as to the meaning of the phrase 'amount of heat.'

II. THERMOMETERS AND CALORIMETERS.

The instruments used for recording the condition of bodies with regard to sensible heat are called thermometers (θερμος, heat; μετρον, a measure). The instruments used for determining the amount of heat in a body are called calorimeters (*calor* heat).

A. Thermometers.

(1) The principle of the thermometer. (2) Its construction. (3) Thermometric scales.

(1) *The principle.*—To ascertain the state of any solid or fluid in connection with sensible heat, we observe its effect upon some liquid. As we know that sensible heat makes liquids (and indeed all bodies) expand, we resolve to use the amount of the expansion of some special liquid when acted upon by the sensible heat of the body under investigation as the record of its temperature. The liquid usually employed is mercury, as its expansion is greater than that of many other liquids, and can be easily observed. For low temperatures alcohol is used, as mercury can be frozen at a not very low temperature. For very high temperatures the expansion of no liquid can be utilized, and the more resistant solid is employed.

(2) *Construction.*—A hard glass tube with a bore so narrow that it is called capillary (*capilla*, a hair) is taken. A bulb is blown at one end. The bulb is then heated. Some of the contained air is thus expelled. The open end of the tube is immersed in mercury. As the bulb cools, the pressure of the air outside forces up the capillary tube some of the mercury. The finger is placed upon the open end of the tube, and the latter is removed from the mercury and turned into its original position, with the bulb downwards. The mercury falls into the bulb. This process of heating, immersion in mercury, reversal is repeated until a sufficient quantity of the liquid is in the instrument. Above the mercury, in the capillary part of the tube, is now air. This is removed by again heating the bulb; the mercury, rising, pushes out all the air, and when the liquid is just at the open end, that open end is sealed in a flame. Now when the mercury sinks back into the bulb, it will have a vacuum, sometimes called the Torricellian vacuum, above it.

(3) *Thermometric scales.*—The next process is to graduate the

thermometer, but this is not effected generally until several weeks after the construction, as the instrument undergoes a certain amount of shrinking that would render any immediate marking slightly inaccurate. After due time has been allowed to elapse, two fixed points are sought. It must be remembered that these fixed points are entirely arbitrary. Any might have been selected. The ones resolved upon are the levels whereat the mercury stands when the thermometer is immersed in melting ice and in the steam of boiling water respectively. For the sake of brevity these two points are called the freezing and the boiling points, but the student is again reminded that they are quite arbitrary, and have no hidden mystic significance. Water is chosen as the commonest liquid, and the freezing point on any thermometer only means the level whereat the mercury or other liquid therein will stand when the instrument is placed in melting ice. When these two fixed points are obtained, the space between is accurately divided into a certain fixed (arbitrarily fixed) number of spaces, called degrees. The scale thus attained is attached to the side of the thermometer, or in the more accurate instruments is marked on the glass by the following process. The tube and bulb are covered with wax. The scale is marked through the wax, so as to expose the glass beneath at places corresponding with the degrees. Then the thermometer is exposed to the vapour of hydrogen fluoride (hydrofluoric acid). This marks the exposed glass. On removal from the vapour and solution of the wax in ether, the instrument is found to be marked off in degrees.

The scales used are three in number—those of Celsius, of Réaumur, of Fahrenheit. The first is more generally known as the Centigrade scale; it is used almost universally upon the Continent, and quite universally by scientific men. The second is used in Russia and some parts of Germany, the third by English-speaking folk. Ultimately, when the dogged Conservatism of Russia and England has given way, both the Réaumur and Fahrenheit scales will disappear, and the Centigrade scale only be used.

(*a*) In the Centigrade scale, freezing point = 0°, boiling point = 100°; spaces between are 100.

(*b*) In the Réaumur scale, freezing point = 0°, boiling point = 80°; spaces between are 80°.

(*c*) In the Fahrenheit scale, freezing point = 32°, boiling point = 212°; spaces between are 180.

It is often requisite to express the temperature upon one of the scales in terms of one or both of the two other scales. Thus a thermometer in a Paris hotel registers 15°, and the English paterfamilias wonders that the day is not bitterly cold, with the temperature 17° below freezing point. His son or daughter should be able, therefore, on emergency, to translate 15° Centigrade into the barbaric Fahrenheit scale. Before the method of doing this is explained, it is desirable again to warn the reader that the numbers used to denote the freezing point, the boiling point, and all intermediate points, are arbitrary. Letters might have been taken instead of numbers, or any other set of numbers might have been employed. That the freezing point is 0° does not mean that there is no heat. That the boiling point is 100° does not mean that the amount of heat is 100 times as much as when the temperature is 1°. To say that a room is at a temperature of 15° does not mean anything more than that the condition of the air in that room in respect to heat perceptible by the senses and able to affect the expansion of mercury in a thermometer, is such that the mercury in such a thermometer, when the instrument is exposed to the air of the room, stands level with the mark 15.

(i.) *To turn a certain number of degrees expressed on the Réaumur scale into the equivalent expression on the Centigrade scale.*

From freezing to boiling point on Réaumur scale = 80 spaces.
,, ,, ,, Centigrade ,, = 100 ,,

80 spaces on the Réaumur scale correspond with 100 on the Centigrade.

1 space on the Réaumur scale corresponds with $\frac{100}{80} = \frac{5}{4}$ on the Centigrade.

If R be the number of degrees on the Réaumur scale, the level of the mercury is R spaces above freezing point, 0°; on the Centigrade the level of the mercury will be $\left(R \times \frac{5}{4}\right)$ spaces above freezing point; and as the freezing point on the Centigrade scale = 0°, $\left(R \times \frac{5}{4}\right)$ will be the number of degrees on the Centigrade that corresponds with R° on the Réaumur. Hence the rule

$$R \times \frac{5}{4} = C.$$

(ii.) *To turn a certain number of degrees expressed on the Centigrade scale into the equivalent expression on the Réaumur scale.*

From freezing to boiling point on the Centigrade scale = 100 spaces.

From freezing to boiling point on the Réaumur scale = 80 spaces.

100 spaces on the Centigrade scale correspond with 80 on the Réaumur.

1 space on the Centigrade scale corresponds with $\frac{80}{100} = \frac{4}{5}$ on the Réaumur.

If C be the number of degrees on the Centigrade scale, the level of the mercury is C spaces above freezing point, 0° ; on the Réaumur scale the level of the mercury is $\left(C \times \frac{4}{5}\right)$ spaces above freezing point; and as the freezing point on the Réaumur scale $= 0°$, $\left(C \times \frac{4}{5}\right)$ will be the number of degrees on the Réaumur scale that corresponds with C° on the Centigrade. Hence the formula

$$C \times \frac{4}{5} = R.$$

(iii.) *Fahrenheit to Centigrade.*

From freezing point to boiling point on Fahrenheit scale = 180 spaces (212 − 32).

From freezing point to boiling point on Centigrade scale = 100 spaces.

180 spaces on the Fahrenheit scale correspond with 100 on the Centigrade.

1 space on the Fahrenheit scale corresponds with $\frac{100}{180} = \frac{5}{9}$ on the Centigrade.

If F be the number of degrees on the Fahrenheit scale, the level of the mercury is $(F - 32)$ spaces above the freezing point, 32° ; on the Centigrade the level of the mercury will be $(F - 32)\frac{5}{9}$ above freezing point; and as on the Centigrade scale that point, $= 0°$, $(F - 32)\frac{5}{9}$ will be the number of degrees on the Centigrade scale that corresponds with F° on the Fahrenheit. Hence the formula

$$(\mathrm{F} - 32)\frac{5}{9} = \mathrm{C}.$$

(iv.) *Centigrade to Fahrenheit.*

As above, 1 space on the Centigrade scale corresponds with $\frac{9}{5}$ space on the Fahrenheit.

If C be the number of degrees on the Centigrade scale, the level of the mercury is C spaces above the freezing point, 0°; on the Fahrenheit scale the level of the mercury will be $\left(\mathrm{C} \times \frac{9}{5}\right)$ above the freezing point, 32°. Hence $\left(\mathrm{C} \times \frac{9}{5}\right) + 32$ will be the number of degrees on the Fahrenheit scale that corresponds with C° on the Centigrade. Hence the formula

$$\left(\mathrm{C} \times \frac{9}{5}\right) + 32 = \mathrm{F}.$$

(v.) *Fahrenheit to Réaumur.*

By reasoning similar to that under (iii.),

$$\left(\mathrm{F} - 32\right)\frac{4}{9} = \mathrm{R}.$$

(vi.) *Réaumur to Fahrenheit.*

By reasoning similar to that under (iv.),

$$\left(\mathrm{R} \times \frac{9}{4}\right) + 32 = \mathrm{F}.$$

Example 127.—Express 15° C. in terms of R. and of F.

100 spaces on the Centigrade scale correspond with 80 on the Réaumur and 180 on the Fahrenheit.

1 space on the Centigrade scale corresponds with $\frac{80}{100}$ or $\frac{4}{5}$ on the Réaumur and $\frac{180}{100}$ or $\frac{9}{5}$ on the Fahrenheit.

15 spaces on the Centigrade scale correspond with $\frac{4}{5} \times 15$ $(= 4 \times 3 = 12)$ on the Réaumur and $\frac{9}{5} \times 15$ $(= 9 \times 3 = 27)$ on the Fahrenheit;

$\therefore$ on R. = 12°. On F. = 27 above freezing point = 27 + 32 = 59°.

Example 128.—Express 20° R. in terms of C. and F.

80 spaces on the Réaumur scale correspond with 100 on the Centigrade and 180 on the Fahrenheit.

1 space on the Réaumur scale corresponds with $\frac{100}{80}$ or $\frac{5}{4}$ on the Centigrade and $\frac{180}{80}$ or $\frac{9}{4}$ on the Fahrenheit.

20 spaces on the Réaumur scale correspond with $\frac{5}{4} \times 20$ ($=5 \times 5=25$) on the Centigrade and $\frac{9}{4} \times 20$ ($=9 \times 5=45$) on the Fahrenheit;

∴ on C. $=25°$; on F. $=45$ above freezing point $=45+32=77°$.

Example 129.—What degree on the Centigrade scale corresponds with $-40°$ F.?

180 spaces on the Fahrenheit scale correspond with 100 on the Centigrade.

1 space on the Fahrenheit scale corresponds with $\frac{100}{180}$ or $\frac{5}{9}$ on the Centigrade.

But $-40°$ F. is 72 spaces below freezing point;

72 spaces on the Fahrenheit scale correspond with $\frac{5}{9} \times 72$ ($=5 \times 8=40$) below freezing point on the Centigrade; and as the freezing point on the Centigrade scale is $0°$, $-40°$ F. $=-40°$ C.

Example 130.—What degree on the Centigrade scale registers the same temperature as five times that number of degrees on the Fahrenheit scale?

Let x = number of degrees C.; then $5x$ = degrees F.,

$$\text{and } x \times \frac{9}{5}+32=5x,$$
$$9x+160=25x,$$
$$160=16x,$$
$$10=x,$$
$$10° \text{ C.}=50° \text{ F.}$$

B. Calorimeters.

As a calorimeter is used to record the amount of heat in a body, it will first be necessary to fix upon a definite unit of heat. It will be impossible to speak of substances containing greater or less quantity of heat than others with any numerical accuracy, unless we have a unit quantity of heat.

Water is the substance most frequently met with. Let us consider the mass of water whose weight is one gram. Suppose such a mass at 0° C. Apply to this mass of water heat until its temperature is raised to 1° C. The amount of heat requisite to effect this is taken as the unit of heat. Hence a thermal unit or calorie, or unit quantity of heat, is that quantity of heat which will raise the mass of one gram of distilled water from 0° to 1° C. It is exceedingly important that the student should have this notion of the thermal unit quite clear. He must observe that it is a quantity of heat, and that quantity which applied to a mass of distilled water whose weight is one gram raises its temperature 1° C.

2	grams	mass	raised from	0° to 1°	would need	2	thermal units.
1	gram	,,	,,	0° to 2°	,,	2	,,
2	grams	,,	,,	0° to 2°	,,	4	,,
10	,,	,,	,,	0° to 5°	,,	50	,,
7½	,,	,,	,,	4° to 8°	,,	30	,,

The number of thermal units required equals the product of the mass and the number of degrees of rise of temperature.

In like manner water cooling gives out thermal units.

1	gram	mass of water	cooling from	1° to 0°	gives out	1	thermal unit.
2	grams	,,	,,	1° to 0°	give out	2	thermal units.
6	,,	,,	,,	20° to 0°	,,	120	,,
3¼	,,	,,	,,	30° to 26°	,,	13	,,

The number of thermal units given out equals the product of the mass and the number of degrees of fall of temperature.

A connection has been established between the thermal unit and mechanical work. Joule established the fact that 772 pounds of mass falling through one foot give out a thermal unit, when unit rise of temperature = 1° F.; or 1 pound falling through 772 feet gives out a thermal unit; or 336 pounds falling through two feet give out a thermal unit.

$$772 \times 1 = 772, \quad 1 \times 772 = 772, \quad 336 \times 2 = 772.$$

The product of feet passed over and pounds of falling mass = 772 whenever a thermal unit is given out. Hence it is usual to say that the mechanical equivalent of heat is 772 foot-pounds.

The expression means that one thermal unit is evolved by 772 foot-pounds of mechanical work.

The mechanical equivalent on the French scale is 424 metre-grams.

In the calorimeter, the quantity of heat in a body is determined by finding how many thermal units it gives out whilst cooling to a known mass of water, or more generally of ice. If a body containing a large quantity of heat is immersed in a known mass of water at a known temperature, and when both the body and the water are at the same temperature that common temperature is noted, the number of degrees rise of temperature of the water is known, and this multiplied by its mass in grams = the number of thermal units given out by the substance under trial. Usually ice and not water is employed, and from the amount of ice melted, the number of thermal units given out can be calculated. As this calculation, however, cannot be understood until latent heat has been studied, we postpone any examples on the ice calorimeter until later.

Example 131.—How many thermal units are required by 40 grams of water when they are to be raised from 10° to $10\frac{1}{4}$°?

1 gram water to be raised through 1° requires 1 thermal unit,

40 grams of water to be raised through $\frac{1}{4}$° require $\frac{1 \times 40}{4} = 10$ thermal units.

Example 132.—60 grams of water cool from a certain temperature to the third of that temperature. In doing so they give out 30 thermal units. What were the initial and final temperatures?

1 gram of water gives out 1 thermal unit in cooling from 1° to 0°,
60 grams „ give out 30 thermal units „ through $\frac{1}{2}$°.

Let x = original temperature.

$$\text{Then } \frac{x}{3} = \text{final, and } x - \frac{x}{3} = \frac{1}{2},\ 6x - 2x = 3,\ x = \frac{3}{4}.$$

Temperatures are $\frac{3}{4}$° and $\frac{1}{4}$°.

Example 133.—A kilogram of a body is placed in a kilogram of water at 10°. The final temperature of the water is 30°. How many thermal units were given out by a gram of the substance?

1 gram of water is raised through 1° by 1 thermal unit,

1000 grams of water are raised through 20° by 1000×20 thermal units.

But this 1000×20 thermal units are given out by 1000 grams of the substance.

$$\therefore \text{ 1 gram of the substance } \frac{1000 \times 20}{1000} = 20.$$

Each gram gives out 20 thermal units.

EXERCISES.

1. Express 12° C., 180° C., −27° C. on the Réaumur and Fahrenheit scales.

2. Express 80° R., $3\frac{3}{7}$ R., −4° R. on the Centigrade and Fahrenheit scales.

3. Express 52° F., 464° F., −12° F. on the Centigrade and Réaumur scales.

4. How many thermal units are evolved by 400 grams of water cooling through $5\frac{1}{4}$°, and by 20 grams cooling from 79·75° to 76·25°?

5. How many thermal units are required to raise the temperature of $\frac{1}{4}$ gram of water from freezing point to 4° C.?

6. The heat given out by 1 kilogram of water in cooling from boiling point to 10° is used to raise a mass of water from 4° to 13°. What is the mass of water?

CHAPTER XXIX.

III. EFFECTS OF HEAT

A. Expansion.

THE most general and noticeable effect of heat upon bodies is the increase of their bulk that results from the application of heat. If any solid, or liquid, or gas is subjected to the action of heat, the solid, liquid, or gas increases in bulk or expands. We have seen ere this, that cohesion is the form of attraction that keeps like particles together, and that cohesion is opposed by heat which tries to separate like particles. This separation of similar particles causes the expansion of the body consisting of such particles.

As heat is opposed by cohesion, and is also the cause of expansion, it is plain that upon bodies wherein cohesion is strongly marked, the expansive effect of heat will be less than it is upon bodies wherein cohesion is weak. Hence solids expand less under the application of heat than liquids do, and liquids again expand less than gases.

To compare the amounts of expansion of different bodies, and of the same body under the action of different quantities of heat, a certain convenient and important term is used. Consider a unit length of any substance at 0°. Let its temperature be raised to 1°. The substance is longer than it was at 0°. The amount of increase of length is called the *coefficient of expansion* for the particular substance. Hence coefficient of linear expansion means the amount of increase in length of a unit of length of the body when its temperature is raised from 0° to 1°. In like manner, the coefficient of cubical expansion means the amount of increase in volume of a unit of volume of a body when its temperature is raised from 0° to 1°.

(1) Expansion in Solids.—As the cohesive force in different solids varies, the coefficients of expansion in solids vary. All solids do not expand equally under equal supplies of heat. The coefficients of linear expansion in solids are very small fractions. *E.g.* that of gold unannealed is ·000015515, of silver ·000019086, of iron ·00001235. It must not be forgotten that the coefficient of expansion is the increase of *unity*, when raised from 0° to 1°. Thus in the case of iron,

1 metre at 0° becomes $1{\cdot}00001235$ at 1°,
" " $1+({\cdot}00001235)\times 2$ at 2°,
" " $1+({\cdot}00001235)\times 10$ at 10°;
4 metres at 0° become $4+({\cdot}00001235)\times 4$ at 1°,
" " $4+({\cdot}00001235)\times 4\times 10$ at 10°.

When the coefficient of linear expansion is known, the cubical may always be taken as approximately three times the linear. For suppose the coefficient of linear expansion to be c in a given solid. Now c is a very small fraction.

A unit of length at 0° becomes $(1+c)$ at 1°,
" volume " $(1+c)^3$ "
" " " $1+3c+3c^2+c^3$.

But if c is a very small fraction, $3c^2$ and $3c^3$ are exceedingly minute, and may be neglected;

$\therefore$ a unit of volume at 0° becomes $(1+3c)$ at 1°;
$\therefore$ $3c=$ coefficient of cubical expansion.

Hence the coefficient of cubical expansion = three times the coefficient of linear expansion.

(2) Expansion in Liquids.—As cohesive force varies in different liquids, the coefficients of expansion in different liquids vary. All liquids do not expand equally under equal supplies of heat. The coefficients of expansion of liquids are greater than those of solids, as might be anticipated from the less amount of cohesion, but they are still small fractions. That of alcohol = ·0010661, that of ether ·0015408.

Three special peculiarities in respect to the expansion of liquids must be dealt with here. (*a*) Care must be taken to distinguish between the apparent and the real expansion of liquids. When a liquid is made to expand in a glass vessel, the glass vessel expands as well as the liquid, and the apparent expansion of the

apparent [illegible] expansion. Real is less than apparent.

liquid is less than the real on account of the expansion of the containing vessel.

(*b*) As the temperature rises, the coefficients of expansion of liquids increase. Thus the amount of expansion of a volume of ether, when its temperature is raised from 0° to 1°, is less than its amount of expansion when its temperature is raised from 20° to 21°.

(*c*) Water is abnormal in its behaviour under heat. If water is cooled down from 100°, it contracts steadily until the temperature of 4° is reached. If the temperature be still lowered through 3°, 2°, 1°, down to the freezing point, instead of contracting, water expands. Again, melted ice or water at 0°, if it has its temperature raised, contracts until 4° is reached. Then it becomes normal again, and as the temperature rises, it expands. Hence if a quantity of water at 4° is either heated or cooled, it expands; hence also water at 4° is at its greatest density. Rise or fall of temperature makes it rarer; 4° is the point of maximum density of water. A valuable result follows from this phenomenon. As the surface water of a pond cools down below 4°, it becomes lighter and floats, and the water freezes on the surface first, forming a non-conducting layer of ice above the unfrozen and warmer water below.

Example 134.—A bar of a metal whose coefficient of expansion is ·00002, has its temperature raised from 0° to 200°. Its original length at 0° is 2 metres 50 centimetres. What is its length at 200°?

A bar of 1 metre becomes a bar of 1 + (·00002 × 200) at 200°,
" 2·5 " " 2·5 + (·004 × 2·5) "
= 2·5 + ·01 = 2·51 "
Elongation is therefore 1 centimetre.

Example 135.—What must be the coefficient of expansion of a substance that a bar 4·0012 metres long, in cooling down from 100° to 0°, may shorten down to a length of 4 metres?

A bar of 1 metre, cooling through 1°, shortens to the extent of that fraction of a metre that is denoted by the coefficient of expansion.

A bar of 4 metres, cooling through 1°, shortens to the extent of that fraction of a metre that is denoted by (4 × the coefficient of expansion).

A bar of 4 metres, cooling through 100°, shortens to the extent of that fraction of a metre that is denoted by 4 × 100 × the coefficient of expansion.

This bar loses ·0012 in cooling through 100°.

Let x be the coefficient of expansion.

$$\text{Then } \cdot0012 = 400x, \text{ and } x = \frac{\cdot0012}{400} = \cdot000003.$$

Or 4 metres whose temperature is raised 100° become 4·0012.

1 metre whose temperature is raised 100° becomes 4·0012 ÷ 4 = 1·0003.

∴ 1 metre whose temperature is raised 100° elongates ·0003 metres.

1 metre whose temperature is raised 1° elongates ·000003 metres, and this is the coefficient of expansion.

Example 136.—10 cubic centimetres of a solid, raised from 5° to 15°, expand to a bulk of 10·012 cubic centimetres. Find the coefficient of linear expansion.

10 c.c. raised through 10° expand ·012,

1 " " 1° expands $\frac{\cdot012}{10 \times 10} = \cdot00012$;

∴ 1 c.c. would have a cubical expansion of ·00012,

and 1 cm. " linear " $\frac{\cdot00012}{3} = \cdot00004$.

Example 137.—A vessel holding at 0°, 5 litres, is filled with a liquid. The temperature of the vessel and contents is raised to 20°. The overflow is 1 decilitre 5 centilitres. What is the coefficient of expansion of the liquid?

We may here disregard the expansion of the vessel.

Cubical expansion of the liquid $= \frac{15}{100}$, or $\frac{3}{20}$ of a litre $= \cdot15$.

5 litres, raised through 20°, expand ·15,

1 litre, " " 1°, expands $\frac{\cdot15}{5 \times 20} = \cdot0015$.

(3) Expansion in Gases.—In gases cohesion is not. The coefficients of expansion are not variable in gases generally. This important fact renders calculations in regard to the volumes of gases under varying temperatures very easy. Experimentally, the coefficient of expansion of ordinary cases is found to be $\cdot 003665$ or $\frac{1}{273}$.

Thus 1 litre of gas at 0° becomes $1\frac{1}{273}$ at 1°,

" " " $1\frac{2}{273}$ at 2°,

" " " $1\frac{10}{273}$ at 10°,

" " " $1\frac{273}{273}$, or 2 at 273°.

2 litres of gas at 0° become $2\frac{2}{273}$ at 1°,

" " " $2\frac{4}{273}$ at 2°,

" " " $2\frac{20}{273}$ at 10°,

" " " $2\frac{2 \times 273}{273}$, or 4 at 273°.

Calculations are most easily made by remembering that

273 volumes of gas at 0° become	274	volumes at	1°,
" " "	275	"	2°,
" " "	283	"	10°,
" " "	546	"	273°,
	$273 + n$	"	$(0 + n)^\circ$

and referring everything to 273 at 0°.

Example 138.—What will be the volume of 455 cubic centimetres of gas at 0° if the temperature is raised to 10°?

273 c.c. at 0° become $(273+10)$ at 10°,
1 ,, becomes $\frac{283}{273}$ at 10°,
455 ,, become $\frac{283 \times 455}{273}$
$$= \frac{283 \times 5}{3} = \frac{1415}{3} = 471\frac{2}{3}.$$

Example 139.—15 cubic centimetres of a gas are measured off at 27°. What volume will they occupy at 0°?

273 at 0° become $(273+27)$ at 27°;
$\therefore$ 300 at 27° become 273 at 0°,
1 ,, becomes $\frac{273}{300}$ at 0°,
15 ,, become $\frac{273}{300} \times 15$
$$= \frac{273}{20} = 13\frac{13}{20}.$$

Example 140.—4 litres of gas at 17° have their temperature raised 60°. What increase of volume occurs?

$(273+17)$ at 17° have a volume of $(273+77)$ at 77°
1 ,, has ,, $\frac{350}{290}$ at 77,
4 ,, have ,, $\frac{350}{290} \times 4$
$$= \frac{35 \times 4}{29} = \frac{140}{29} = 4\frac{24}{29};$$
$\therefore$ increase of volume $= 4\frac{24}{29} - 4 = \frac{24}{29}$ litre.

Example 141.—82 cubic centimetres of hydrogen at 55° are cooled down until they occupy a volume of 70 cubic centimetres. What is now the temperature?

Let x = number of degrees of final temperature.
$273+55$ or 328 at 55° become $273+x$ at x°,
1 ,, becomes $\frac{273+x}{328}$ at x°,
82 ,, become $\frac{(273+x) \times 82}{328}$
$$= \frac{273+x}{4}.$$

$$\text{But } \frac{273+x}{4}=70;\ \therefore\ 273+x=280,$$
$$x=280-273$$
$$=7^\circ.$$

The above law as to the uniformity of the coefficient of expansion in gases is the law of Gay-Lussac. Most frequently this law has to be used in combination with Boyle's law (see p. 155). In the chemical laboratory many calculations have to be made as to gases. Now, to compare gases, they must always be at the same temperature and under the same pressure, as the least alteration in either temperature or pressure makes a difference in the volume of a gas. Chemists have therefore agreed always to compare gases at a particular temperature and under a particular pressure. The temperature, arbitrarily but conveniently chosen, is 0° C.; the pressure is one equivalent to that of a column of mercury 760 mm. high. When gases are at this temperature and under this pressure, they are said to be at normal temperature and pressure; hence, in short phraseology, a gas at 0° and 760 mm. is at normal.

It is manifest, however, that upon very rare occasions will gases made in a laboratory be at normal. The temperature is usually higher than 0°, and the atmospheric pressure in England is very variable. It is matter therefore of the utmost importance to be able to translate the volume of a gas when measured off at any temperature or under any pressure into its equivalent at 0° and under 760 millimetres. This can be done by aid of Gay-Lussac's and Boyle's laws. These statements will now be repeated and a few examples worked.

Gay-Lussac's law.—Gases expand $\frac{1}{273}$ of their volume at 0° for every degree rise of temperature.

Boyle's.—The volume of a gas is inversely as the pressure upon it.

Example 142.—13 litres of a gas at normal have their temperature raised to 21° and the pressure increased to 840 millimetres. What is now the volume of the gas?

273 at 0° under pressure of 760 mm. become 294 at 21°.

1 ,, 1 mm. becomes $\frac{294}{273}\times 760$ at 21°.

13 ,, 840 mm. become $\frac{294\times 760\times 13}{273\times 840}$

$$=\frac{38}{3}=12\tfrac{2}{3}.$$

Example 143.—$17\frac{7}{9}$ cubic centimetres of oxygen at − 13 and under pressure 570 millimetres. What volume will they occupy at normal?

260 at − 13° under pressure of 570 mm. become 273 at 0°.

1 ,, ,, 1 mm. becomes $\frac{273 \times 570}{260}$ at 0°.

$17\frac{7}{9}$,, ,, 760 mm. become $\frac{273 \times 570 \times 160}{260 \times 760 \times 9}$

$$= \frac{7 \times 190 \times 2}{190}$$

$$= 7 \times 2 = 14.$$

Example 144.—What change of volume will 5 litres of gas undergo in being raised from − 23° through 100°, whilst the original pressure upon them of 800 millimetres is reduced $\frac{1}{8}$?

250 at − 23° and under pressure of 800 mm. become 350 at 77°.

1 ,, ,, 1 mm. becomes $\frac{350 \times 800}{250}$ at 77°.

5 ,, ,, 700 mm. become $\frac{350 \times 800 \times 5}{250 \times 700}$

$$= \frac{7 \times 8}{7} = 8;$$

∴ change of volume = 8 − 5 = 3 litres.

Example 145.—20 cubic centimetres of nitrogen are obtained at normal temperature and pressure. The pressure upon them is now doubled. To what temperature must they be raised that no alteration of volume may occur?

20 under pressure of 760 become 10 under double the pressure or 1520 mm.

273 at 0° become 546 at 273°.

1 at 0° becomes $\frac{546}{273}$ or 2 at 273°.

10 at 0° become 20 ,,

∴ required temperature is 273°.

B. Change of State.

If heat is applied to a solid, the solid expands. Continue the application of the heat a sufficient length of time, and the expansion of the solid becomes so great that the solid passes into the liquid condition. This is the process of liquefaction.

If heat is applied to a liquid, the liquid expands. Continue the application of the heat a sufficient length of time, and the expansion of the liquid becomes so great that the liquid passes into the gaseous condition. This is the process of vaporization.

(1) *Liquefaction.*—Let us investigate a special case. Heat is applied to a piece of ice at 0°. The ice begins to melt. It melts very slowly, even though large quantities of heat are brought to it. And during all the time that it is melting, its temperature, as registered by a thermometer, is 0°. The temperature never rises above 0° until all the ice is melted into water. Then, and not until then, the temperature of the water begins to rise. What then is becoming of all the quantity of heat poured into the ice? It is becoming *latent* or hidden: it is ceasing to be sensible to the thermometer. This latent heat of ice can be made sensible again; for when the water is turned back again into ice by a reduction of temperature, the latent heat is evolved again, and becomes sensible and capable of affecting a thermometer.

Not only is this curious temporary disappearance of heat noticed in the case of the liquefaction of ice; it is noticed in all cases of liquefaction of solids. Whenever any solid passes into the liquid state, a certain amount of heat becomes latent; and when the liquid repasses into the solid state again, that same amount of heat reappears in the sensible condition. This quantity of heat vanishing as a solid becomes a liquid, appearing again as the liquid becomes a solid, is called the *latent heat of liquefaction.*

From the foregoing paragraphs the student will have gathered that the amount of heat thus becoming latent is definite. Every liquid has a definite latent heat. Taking again the case of the conversion of ice into water: if a gram of ice at 0° is mixed with 79 grams of water at 1°, two grams of water at 0° result; or if the gram of ice at 0° is mixed with 1 gram of water at 79°, two grams of water at 0° result. In each of these cases 79 × 1 or 1 × 79 thermal units disappear as 1 gram of ice at 0° becomes one gram of water at 0°. Hence the latent heat of water is said to be 79 or roughly 80. The expression means that whenever a gram of ice at 0° is changed into water without any rise of temperature, 79 thermal units become latent.

Two grams of ice would need 2 × 80 or 160 thermal units if melted without any rise of temperature.

Every liquid has its own special latent heat. The numbers of thermal units becoming latent vary, but are generally far less than 80.

Example 146.—How many grams of water at 100° would be needed for the complete conversion of one kilogram of ice at 0° into water at 0°?

1 gram of ice at 0° requires 80 thermal units to melt it.
1000 grams „ require 80×1000 thermal units to melt them.
100 thermal units are given out by 1 gram cooling from 100° to 0°.
1 thermal unit is „ $\frac{1}{100}$ „ „
80,000 thermal units are „ $\frac{1}{100} \times 80{,}000 = 800$ grams in cooling from 100° to 0°.

Example 147.—2 grams of water at 20° are mixed with ice. How much ice is melted?

1 gram of water at 20° gives out 20 thermal units in passing from 20° to 0°.

2 grams of water at 20° give out 40 thermal units in passing from 20° to 0°.

1 gram of ice requires 80 thermal units to melt; ∴ $\frac{1}{2}$ gram of ice will be melted.

Example 148.—$\frac{1}{2}$ gram of ice is to be melted and the resulting water raised to 20°. What mass of water at 25° would effect this?

1 gram of ice requires 80 thermal units to melt, and 20 thermal units to raise the resulting water to 20°.

$\frac{1}{2}$ gram of ice requires 40 thermal units to melt, and 10 thermal units to raise the resulting water to 20° = 50 thermal units required.

Water, in passing from 25° to 20°, gives out 5 thermal units per gram.

50 thermal units are required; ∴ $\frac{50}{5} = 10$ grams of water must be used.

Example 149.—$10\frac{1}{2}$ grams of water at boiling point are mixed with a certain mass of ice, and the resulting temperature is that at which the maximum density of water is attained. What was the mass of the ice?

The maximum density of water is attained at 4°.

1 gram of water at 100° gives out 96 thermal units in passing to 4°.

$10\frac{1}{2}$ grams of water at 100° give out $96 \times 10\frac{1}{2}$ or 1008 thermal units in passing to 4°.

1 gram of ice requires 80 thermal units to melt, and 4 thermal units to raise to 4°; 1 gram, therefore, will require 84 thermal units.

Let x be the number of grams of ice.

$$\text{Then } x = \frac{1008}{84} = \frac{84}{7} = 12 \text{ grams.}$$

Example 150.—$27\frac{1}{2}$ grams of water at 15° are mixed with 25 centigrams of ice. Find the resulting temperature.

1 gram of water at 15° will give out 15 thermal units in passing from 15° to 0°.

$27\frac{1}{2}$ grams of water at 15° will give out $15 \times 27\frac{1}{2}$ or $412\frac{1}{2}$ thermal units in passing from 15° to 0°.

25 centigrams $= \frac{1}{4}$ gram of ice require 20 thermal units to melt; therefore there are $412\frac{1}{2} - 20 = 392\frac{1}{2}$ thermal units to raise temperature of $27\frac{3}{4}$ grams of water at 0°.

Let x be the final temperature.

$$\text{Then } x = \frac{392\frac{1}{2}}{27\frac{3}{4}} = \frac{\frac{785}{2}}{\frac{111}{4}} = \frac{785}{2} \times \frac{4}{111}$$

$$= \frac{1570}{111} = 14\frac{16}{111}^{\circ}.$$

Or, let $x =$ number of degrees of final temperature.

Then the $27\frac{1}{2}$ grams of water fall through $(15 - x)^{\circ}$, and the water resulting from the melting of the ice rises through x degrees;

$$\therefore 27\tfrac{1}{2}(15 - x) = \tfrac{1}{4}(80 + x),$$

$$\frac{55}{2} \times 15 - \frac{55x}{2} = 20 + \frac{x}{4},$$

$$1650 - 110x = 80 + x,$$

$$1570 = 111x,$$

$$x = 14\frac{16}{111}.$$

Example 151.—3 grams of a solid are melted without any rise of temperature by admixture with them of 2 grams of water at 30°. The melting point and final temperature are 15°. Find the latent heat of the solid.

2 grams of water at 30° give out 30 thermal units in passing from 30° to 15°, *i.e.* through 15°.

30 thermal units melt 3 grams of the solid; therefore 10 thermal units melt 1 gram.

Now 80 thermal units are required to melt 1 gram of ice, the standard, ∴ latent heat of solid to latent heat of water is as 10 to $80 = 1 : 8 = \frac{1}{8}$.

(2) *Vaporization.*—Take a special case again. Heat is applied to a mass of water at 100°. The water begins to pass off as steam. It vaporizes very slowly even though large quantities of heat are brought to it. And during all the time that it is vaporizing, its temperature as registered by a thermometer is 100°

The temperature never rises above 100° until all the water is vaporized into steam. Then, and not until then, the temperature of the steam begins to rise. Heat has become latent. This heat, non-sensible to the thermometer, can be made evident again by re-turning the steam into water. Then the heat that had become latent is again evolved.

This disappearance of heat occurs whenever any liquid becomes a gas, and reappears to the same amount whenever the gas returns to the liquid state. And this quantity of heat vanishing as a liquid becomes a gas, and reappearing as a gas becomes a liquid, is called the latent heat of vaporization.

The amount again is definite in every liquid. The latent heat of steam *at 100°* is 537—*i.e.*, when a gram of water at 100° becomes a gram of steam at 100°, 537 thermal units, or enough heat to raise 537 grams of water from 0° to 1°, disappear. It will be noticed that the words 'at 100°' are italicised, and this is because water passes off as vapour at any temperature, not at 100° only. Thus the water of a pond on a cold day is passing off as vapour even though the air temperature is far below 100°. This statement leads to the distinction between (*a*) evaporation, (*b*) ebullition or boiling.

(*a*) Evaporation.—When a liquid passes off into the gas form at any indefinite temperature, it is evaporating. Evaporation is the noiseless, imperceptible, slow, constant passage of liquid into gas. Its amount is affected by the extent of liquid surface exposed, by the pressure of the air upon the liquid, by the condition of the air in relation to steam or other gases, by the stillness or want of stillness of the air, by the temperature.

(i.) The greater the surface of liquid exposed, the more evaporating space is there; for only the surface of the liquid evaporates.

(ii.) The greater the pressure of the air upon the liquid, the less will be the evaporation, as the gas passing off has to struggle against the air pressure.

(iii.) The greater the amount of steam or other gas already in the air, the less will be the evaporation of the liquid, as there will be difficulty on the part of its corresponding gas in passing into a space already charged with that gas. Thus water evaporates with difficulty into an atmosphere charged with steam, ether evaporates with difficulty into an atmosphere charged with vapour of ether.

(iv.) The greater the stillness of the air, the less the evaporation. If the air is still, the part of it in contact with the evaporating liquid is now charged with the vapour, and can take up no more. Then evaporation must cease. But if the air is moving about, the charged air is removed and a fresh quantity free from the gas comes into contact with the liquid, and can take up a fresh supply of its vapour.

(v.) It appears that approximately the same sum is always obtained by adding together the number of degrees at which a liquid becomes a gas, and the number of thermal units that become latent during the process. Water at 100° requires 537 thermal units. $100 + 537 = 637$. Water at 90° requires 544 thermal units. $90 + 544 = 634$. Hence the higher the temperature of the water or other liquid, the less latent heat is required, and the more rapid will be the evaporation.

Hence the greater the liquid surface
,, the less the air pressure
,, the less the amount of the vapour of the liquid in the air
,, the greater the amount of movement of the air
,, the higher the temperature
the more evaporation.

(*b*) Ebullition or boiling.—This is the noisy, perceptible, rapid, occasional passage of liquid into gas. It only takes place at a definite temperature for each particular liquid. It is not as evaporation is, a surface action only. Bubbles of vapour are formed within the liquid, and struggle up to escape at the top. The elastic force of these vapour bubbles is equal to the air pressure at the time.

It has been said that ebullition of a liquid takes place at a definite temperature. This is true in the main. Speaking generally, each liquid has a definite boiling point. But the boiling point may be made to vary by three things.

(i.) The pressure of the air. If ebullition is the formation of vapour bubbles whose elastic force is equal to the air pressure at the time, the greater that pressure, the harder work is it for the liquid to throw off its vapour, and therefore the higher will be the temperature required. With increased air pressure, the boiling point is raised. On the other hand, if the pressure of the air lessens, the easier is it for the liquid to throw off its vapour, and the lower will be the temperature required. With lessened air pressure, the boiling point is lowered. High up on a mountain, water boils at temperatures below 100°, so much below at times that the boiling water does not cook the meat. A rise of about 300 metres above the sea level is accompanied by a lowering of the boiling point 1° C. A rise of 596 feet is accompanied by a lowering of the boiling point 1° F.

(ii.) The presence or absence of salts in solution. If a liquid has any salts dissolved in it, adhesion comes into play. There is adhesion between the particles of the liquid and the particles of the salts. This adhesion opposes ebullition. The liquid is trying to pass away as gas: the salts are trying to hold it back. Hence the presence of salts in solution makes ebullition more difficult, and raises the boiling point. Distilled water, free from salts, boils at a lower temperature than ordinary water, rich in salts.

(iii.) The nature of the vessel. If the vessel wherein the liquid is placed is of such nature as to adhere to the liquid, reasoning similar to the above comes into play, and the boiling point is raised. Water boils at a higher temperature in a greasy vessel than in a clean one.

$$\left.\begin{array}{l}\text{The greater the air pressure}\\ \text{The greater the amount of salts in solution}\\ \text{The greater the adhesion to the containing vessel}\end{array}\right\}\begin{array}{l}\text{the higher}\\ \text{the boiling}\\ \text{point.}\end{array}$$

C. Specific Heat.

(1) *Explanation and definition.*—If a kilogram of water at 0° and a kilogram of mercury at 0° were both subjected to the same source of heat steadily applied, the temperature of both the water and the mercury would rise, but that of the mercury would rise very much more rapidly than that of the water. A thermometer plunged in the mercury would register 100° when a thermometer plunged in the water would only register between 3° and 4°.

Here then are two equal masses of two different liquids subjected to exactly the same source of heat, but rising in temperature at very different rates. The water wants a far greater quantity of heat to raise it through a certain range of temperature than does the mercury. Indeed, a given mass of water needs a greater quantity of heat than an equal mass of anything else known to raise it through a given temperature.

We already have a name for the quantity of heat needed to raise the mass of water whose weight is one gram through 1°. That name is a thermal unit. The quantity of heat needed to raise the mass of any other body whose weight is one gram through 1° will always be less than a thermal unit. The ratio of this quantity to a thermal unit is called the specific heat of the body.

It is the capacity of the body for heat as compared with the capacity of water for heat.

The specific heat of a substance is therefore the ratio to a thermal unit of the quantity of heat required to raise one gram mass of the substance from 0° to 1°.

(2) *Determination of specific heats.* — Three methods are employed to find out the specific heats of bodies.—(*a*) *Rate of cooling.*—If a body rise through a given range of temperature under the action of a small amount of heat, it will fall through the same range of temperature quickly. If a body rise through a given range of temperature under the action of a large amount of heat, it will fall through the same range of temperature comparatively slowly. The water and the mercury of our first paragraph under this head take respectively larger and smaller quantities of heat to be raised from 0° to 100°; the water and the mercury will take respectively longer and shorter times to get rid of the absorbed heat and to cool from 100° to 0°. The rates of cooling will evidently be as the specific heats.

Example 152.—If a kilogram of a liquid took half an hour to cool from 95° to 15°, and 5 hectograms of water took 5 minutes to cool from $8\frac{8}{9}°$ to 0°, what is the specific heat of the liquid?

1000 grams of a liquid cool through 80° in 30′.

1 gram ,, ,, ,, $1°$ in $\frac{30}{1000 \times 80} = \frac{3}{8000}$.

500 grams of water cool through $8\frac{8}{9}°$ in 5′.

1 gram ,, ,, $1°$ in $\frac{5 \times 9}{500 \times 80} = \frac{9}{8000}$;

$$\therefore \text{ specific heats are as } \frac{3}{8000} : \frac{9}{8000}.$$

Taking specific heat of water as 1, the specific heat of the liquid is $= \frac{1}{3}$.

(*b*) *Fusion of ice.*—A gram of ice melting without rise of temperature requires 80 thermal units. If a known mass of a substance at known temperature be placed in the calorimeter referred to on page 246, and surrounded by ice that is itself encased in a non-conducting coating, or better if the calorimeter be made of ice, the substance in cooling down to the temperature of the ice or to 0° will melt a certain quantity of the ice. From the amount thus melted, the number of thermal units given out by the known mass in cooling through a given number of degrees of temperature can be calculated, and hence the fractions of a thermal unit given out by a gram mass of the substance in cooling through 1°. This last amount of heat will be the specific heat.

Example 153.—400 grammes of a solid at 200° are placed in the ice calorimeter, and after it has cooled to 0°, 150 grams of ice are found to be melted. Find specific heat of solid.

400 grams cooling through 200° melt 150 grams of ice.

∴ 400 „ „ „ „ give out 150 × 80 thermal units.

1 gram „ „ „ 1° gives out $\frac{150 \times 80}{400 \times 200}$ „

$$\therefore \text{specific heat of substance} = \frac{\overset{3\cancel{0}}{\cancel{15}\cancel{0}} \times \cancel{8}\cancel{0}}{\underset{5}{\cancel{4}\cancel{0}\cancel{0}} \times 20\cancel{0}} = \frac{3}{20}.$$

Example 154.—400 grams of antimony at 40° are placed in the ice calorimeter; 2 grams of ice are melted. What is the specific heat of antimony?

400 grams of antimony cooling through 40° melt 2 grams of ice.

∴ 400 grams of antimony cooling through 40° give out 2 × 80 thermal units.

1 gram of antimony cooling through 1° gives out $\frac{2 \times 80}{400 \times 40}$;

$$\therefore \text{specific heat of antimony} = \frac{1}{100}.$$

Example 155.—What weight of a body of specific heat $\frac{1}{10}$ must be used that when at an original temperature of 80° it may melt $\frac{1}{2}$ kilogram of ice?

500 grams of ice would require 500 × 80 or 40,000 thermal units to melt it.

1 gram of a body of specific heat $\frac{1}{10}$ gives out $\frac{1}{10}$ thermal unit for every degree through which it falls.

∴ 1 gram of a body of specific heat $\frac{1}{10}$ gives out $\frac{1}{10} \times 80 = 8$ thermal units in passing from 80° to 0°.

40,000 thermal units are required; ∴ $\frac{40000}{8} = 5000$ grams, or 5 kilograms of the substance would be required.

Example 156.—If 3 kilograms of a body of specific heat $\frac{11}{100}$

at a temperature of $\frac{16^\circ}{33}$ are placed in the ice calorimeter, what mass of ice will be melted?

1 gram of the body in passing through 1° gives out $\frac{11}{100}$ thermal unit.

1 gram of the body in passing through $\frac{16^\circ}{33}$ gives out $\frac{11}{100} \times \frac{16}{33}$ thermal unit $= \frac{16}{300} = \frac{4}{75}$ thermal unit.

3000 grams therefore give out $3000 \times \frac{4}{75} = 40 \times 4 = 160$ thermal units.

1 gram of ice requires 80 thermal units to melt;

$\therefore$ 160 thermal units will melt 2 grams of ice.

Now that this method of determining specific heat has been studied, it is possible to combine the calculations as to latent and specific heat. If, for example, we had to determine how many grams of water at 100° were required to melt 4 grams of ice and raise them to a temperature of 20°, the method would be as follows:—

Example 157.—4 grams of ice require $4 \times 80 = 320$ thermal units to melt them, and $4 \times 20 = 80$ to raise them when melted to 20°. In all, $320 + 80 = 400$ thermal units.

Let x be the number of grams of water used. As they are cooled from 100° to 20° or 80°, the thermal units they give out $= 80x$.

$$\therefore 400 = 80x.$$
$$\therefore x = 5 \text{ grams.}$$

Example 158.—3 grams of ice at 0° are mixed with 42 grams of water at 70°. Find the final temperature of the mixture.

3 grams of ice require $3 \times 80 = 240$ thermal units to melt.

42 grams of water in passing from 70° to 0° give out $42 \times 70 = 2940$ thermal units; of these 240 are required to melt the ice, leaving $2940 - 240 = 2700$ thermal units available to raise from 0° the whole 45 grams of water at 0°.

Let x be the final temperature; then $x = \frac{2700}{45} = \frac{540}{9} = 60^\circ$.

Or let x = number of degrees of final temperature.

Then the water cools through $(70-x)^\circ$, and the water resulting from the melted ice rises through x degrees.

$$\begin{aligned} \text{And } 42(70-x) &= 3(80+x). \\ 14(70-x) &= 80+x. \\ 980-14x &= 80+x. \\ 900 &= 15x. \\ x &= 60^\circ. \end{aligned}$$

Example 159.—10 grams of a body whose melting point is 80°, latent heat 20, specific heat $\frac{1}{100}$, temperature 80°, are mixed with water at 84°. The final temperature is 82°. What was the mass of the water?

10 grams of the body require $10 \times 20 = 200$ thermal units to melt, and $10 \times 2 \times \frac{1}{100} = \frac{1}{5}$ thermal unit to raise to 82°;

∴ $200\frac{1}{5}$ thermal units are required to melt and raise the body to the final temperature.

Let x be the number of grams of water; as it cools from 84° to 82°, or through 2°, it gives out $2x$ thermal units;

∴ $2x = 200\frac{1}{5}$, $x = \frac{1001}{5} \times \frac{1}{2} = \frac{1001}{10} = 100\frac{1}{10}$ grams of water.

(*c*) *Method of mixtures.*—If a gram of water at 100° were mixed with a gram of water at 0°, the result would be two grams of water at the average temperature between 100° and 0°, *i.e.* at 50°.

If a gram of mercury at 100° were mixed with a gram of mercury at 0°, the temperature of the two grams would be also 50°.

But if a gram of water at 100° were mixed with a gram of mercury at 0°, the resulting temperature is about 97°. Here then the water of great specific heat has fallen in temperature only about 3°, while the mercury of low specific heat has risen in temperature 97°. We can calculate the number of thermal units given out by the water. It is three thermal units.

1 gram of mercury is raised through 97° by 3 thermal units.

1 " " 1° by $\frac{3}{97}$ "

∴ $\frac{3}{97}$ is the specific heat of mercury.

And in like manner by the method of mixtures the specific heat of many liquids and solids can be ascertained.

Example 160.—10 grams of benzine at 2° are mixed with 2·4 grams of water at 10°. The resulting temperature is 8°. Find the specific heat of benzine.

2·4 grams of water in passing from 10° to 8°, or through 2°, give out 4·8 thermal units.

10 grams of benzine are raised from 2° to 8°, or through 6°, by 4·8 thermal units.

1 gram of benzine is raised through 1°, by $\frac{4{\cdot}8}{10 \times 6} = \frac{{\cdot}4}{5} = \frac{2}{25}$;

$$\therefore \frac{2}{25} \text{ is the specific heat of benzine.}$$

Example 161.—If 8 grams of water at 100° mixed with 11 grams of a solid at 11° give as resulting temperature 99°, what is the specific heat of the solid?

8 grams of water passing from 100° to 99°, or through 1°, give out 8 thermal units.

11 grams of the solid are raised from 11° to 99°, or through 88° by 8 thermal units.

1 gram of the solid is raised through 1° by $\frac{8}{11 \times 88} = \frac{1}{121}$;

$$\therefore \frac{1}{121} \text{ is the specific heat of the solid.}$$

EXERCISES.

1. Find the coefficient of expansion of a solid 2 metres of which at 0° become 2·003815 metres at 100°.

2. What will be the length of a bar of cast iron at 300° if its length at 0° is 9·9630867, and if the coefficient of expansion of cast iron is ·00001235?

3. The coefficient of linear expansion of lead is ·000088483. A bar $\frac{1}{2}$ metre long, 1 decimetre broad, and 5 centimetres high is raised from 0° to 80°. Find the volume it now occupies.

4. A litre of wood spirit at 0° is raised to 10°. 12·02 c.c. overflow. Find the coefficient of expansion of wood spirit.

5. Find the volume of 8 litres of a gas at 0° and 760 mm. when the temperature is raised to 20°.

6. 1 litre of gas is measured off at 13° C., and under a pressure = 78 cm. of mercury. Calculate the volume at normal.

7. 50 c.c. of gas are obtained at 91° C. What will be the change of volume they undergo if the temperature is lowered $45\frac{1}{2}$° and the pressure is doubled?

8. If 4 grams of a solid in melting absorb the heat given out by 40 grams of water cooling from 28° to 26°, find its latent heat.

9. A solid of latent heat 10°, and whose melting point is 10°, is mixed with water at 25°. If the mass of the solid was 6 grams, what is that of the water, final temperature being 10°?

10. 5 parts by mass of water at 80° are mixed with 10 parts by mass of a liquid at 8°. Resulting temperature 78°. What is the specific heat of the liquid?

11. A body of specific heat $\frac{1}{10}$ is mixed mass for mass with water at 100°. What must be the original temperature of the solid that the final temperature may be 95°?

12. 40 grams of a solid at 12°, whose melting point is 12°, are mixed with 95 grams of water at 24°. Result is complete melting and final temperature of 16°. 10 grams of the same solid at 12° are mixed with 10 grams of water at 57°. Resulting temperature = 37°. Find the latent and specific heat of the solid.

CHAPTER XXX.

IV. DIFFUSION OF HEAT.

If a poker be placed in the fire, the end remote from the fire is presently so hot that the hand can with difficulty be maintained in contact with it.

If heat be applied to the bottom of a vessel full of water, the water becomes heated, not only as to its lower layers, but throughout its mass.

If a red-hot cannon ball is brought into a room, every object in that room undergoes a rise of temperature.

These three cases illustrate the three methods whereby heat is diffused.

A. Conduction.

(1) *Definition.*—The first case is a case of conduction. The heat of the fire is said to be conducted along the poker to the distant end. *Conduction of heat is the transmission of heat from particle to particle of a body.* This conduction is gradual, and does not necessarily affect a rectilinear path; for conduction takes place as rapidly along a bent as along a straight bar.

(2) *Conductivity.*—Different substances have different powers for the conduction of heat. Solids are the best conductors. Fluids transmit heat more readily by the method of convection to be presently studied, and conduct heat badly. Of solids, the best conductors are metals, and of these silver is the best, and copper is the next best conductor. These varying powers of transmitting heat from particle to particle are called the conductivities of the substances.

To compare the conductivities of different substances. Suppose a square centimetre of a substance one centimetre in thickness. Suppose the temperature on one side of this square plate of the substance is 0°, and the temperature on the other side is 1°. Heat will flow across the substance from the side at a temperature of 1° to the side at a temperature of 0°. Varying quantities of heat will flow across with varying substances. The quantity that under the above circumstances flows across in 1″ is taken as the measure of the conductivity of the substance. Hence the definition of conductivity is the amount of heat that flows through one centimetre thickness (cross section, one square centimetre) in one second when the difference of temperature of opposite sides is 1°.

Example 162.—If through a cubical piece of a body whose length is 3 centimetres and cross section 16 square centimetres flows in half an hour enough heat to melt 3 grams of ice, the temperature of the side remote from the ice being 5°, and through a cubical piece of another substance 5 centimetres long and 2 centimetres by 7 in section flows in 10 minutes enough heat to melt $\frac{1}{3}$ gram of ice, the temperature of the side remote from the ice being 4°: compare the conductivities of the two bodies.

(i.) With length 3, cross section 16, in $(30 \times 60)''$ difference of temperature 5°, 3 grams ice are melted, or 240 thermal units pass.

With length 1, cross section 1, in 1″, difference of temperature 1° pass $\frac{240 \times 3}{16 \times 30 \times 60 \times 5} = \frac{1}{200}$ thermal unit.

(ii.) With length 5, cross section 14, in $(10 \times 60)''$, difference of temperature 4°, $\frac{1}{3}$ gram ice melted, or $\frac{80}{3}$ thermal units pass.

With length 1, cross section 1, in 1″, difference of temperature 1°, pass $\frac{80 \times 5}{3 \times 14 \times 10 \times 60 \times 4} = \frac{1}{252}$ thermal unit;

$$\therefore \text{ the conductivities are as } \frac{1}{200} : \frac{1}{252}$$
$$= \frac{1}{50} : \frac{1}{63}, \text{ as } 63 : 50.$$

Example 163.—Through a regular cube containing 27 c.c. of a body whose conductivity is represented by 100, flows in 5 minutes enough heat to melt $\frac{1}{2}$ gram of ice, the temperature on the side remote from the ice being 4°. How much ice would be melted in one hour by heat traversing a cubical piece of a body of conductivity represented by 75 when the temperature on the remote side is 12°, and the dimensions are, length 6 cm., cross section 4 sq. cm.?

With thermal conductivity 100, length 3, cross section 9, in $(5 \times 60)''$, difference of temperature 4°, $\frac{1}{2}$ gram ice melted, or 40 thermal units pass.

With thermal conductivity 1, length 1, cross section 1, in 1″, difference of temperature 1°, pass $\frac{40 \times 3}{100 \times 9 \times 5 \times 60 \times 4} = \frac{1}{9000}$.

With thermal conductivity 75, length 6, cross section 4, in

(60 × 60)″, difference of temperature 12°, pass

$$\frac{1 \times 75 \times 4 \times 60 \times 60 \times 12}{9000 \times 6} = 16 \times 15 = 240.$$

$$\text{Number of grams of ice melted} = \frac{240}{80} = 3 \text{ grams.}$$

B. Convection.

The second case is one of convection. When heat is applied to the bottom of a vessel containing water or any other liquid, the amount of conduction of heat though the substance of the liquid is but little. But the heat conducted through the material of the vessel, and conducted to the lowest and outermost layers of water, *i.e.* to the water layers in contact with the bottom and sides of the vessel, raises the temperature of the layers of water thus in contact with the vessel. These layers under the action of the heat expand, become lighter, rise to the upper part of the vessel, whilst other colder, denser, heavier layers fall down and take their place. These latter in their turn are heated through the medium of conduction, expand, become lighter, rise. Other colder layers take their place, and thus a circulation of liquid layers is set up, the heated, lighter layers rising, and the cooler, heavier layers falling. Thus the temperature of the whole mass is raised, and this method is called Convection.

Convection, therefore, is the diffusion of heat by means of the movement of the particles of fluid bodies. In convection there is no transference of heat from particle to particle, but successive layers of fluid come within the range of the heat and are heated.

Convection explains many natural phenomena. (1) For example, the trade winds depend upon the principle of convection. The air in contact with the earth in the tropical regions is heated, becomes lighter, rises. The cold polar air heavier than this tropical air flows southwards from the north pole and northwards from the south pole, in each case tropicwards, to take the place of the uprising tropical air. Hence a current of cold air from the north and one from the south. That the trade winds do not blow directly south or north results from the rotation of the earth upon its own axis. Air coming from the north pole has either no movement in the direction of the earth from west to east, or very little of such movement. Hence, as the earth at the tropics is swinging round from west to east at a great pace, and the air flowing from the poles is going very slowly in that direction, when that air reaches the tropics it appears to be moving in a direction opposite to that of the earth. All movement is relative, and the tropical lands are travelling rapidly from west to east: the polar air is travelling slowly from west to east. Therefore the slow polar air seems to move in regard to the swift earth from east to west, and the trade winds from the north appear to blow from the north-east. Those from the south pole appear to blow from the south-west. The earth moving in a certain direction catches up the air moving more slowly in the same direction. The air appears to blow in a direction opposed to the earth's movement on its own axis.

(2) Land and sea breezes depend upon the principle of convection. During the day the earth absorbs heat. It becomes hot. The air in contact with it becomes hot, lighter, rises, and the cold sea air flows in to take its place. There is a sea breeze. During the night the earth radiates heat. It becomes cool. The air in contact with it becomes cool. The air in contact with the sea, not so rapidly cooled, as water does not radiate heat so rapidly as land, is warmer, lighter, rises. The colder land air flows out to take its place. There is a land breeze.

(3) The method of heating houses by hot-water pipes or hot-air pipes is founded on the principles of convection. The air or water heated below at the furnace becomes expanded, lighter, and rises; whilst colder, heavier air sinks down to take its place and to become heated, expanded, lighter, and rise in its turn.

C. Radiation.

The third case is one of radiation. The cannon ball radiates heat to all the bodies around. There is in this case, or in the case of the sun radiating heat to the planets of its system, nothing of conduction nor of convection. A new method of heat-transmission, not taking place by transmission from particle to particle as in conduction, nor by movement of particles as in convection, presents itself. This method is radiation. Radiation is the passage of heat through space filled with ether. It requires no solid bodies, no liquid bodies. None such intervenes between the sun and us. Radiation requires only ether.

One or two facts as to this method of heat-transmission:—

(1) Heat radiates in right lines like light, and like light undergoes reflection and refraction.

(2) All bodies are persistently radiating heat. Some, however, radiate more than others, and the universal tendency is to bring

all bodies to a state of equilibrium as regards temperature. Thus the cannon ball in our case on page 269 radiates heat, but the various bodies in the room also radiate heat, only in less amount.

(3) The tendency is to equilibrium of temperature of all things. The cannon ball radiates more heat: the objects of the room radiate less heat. By degrees the cannon ball grows cooler, the objects in the room grow more hot. Ultimately, if no disturbing force intervene, the cannon ball and the objects in the room will have the same temperature.

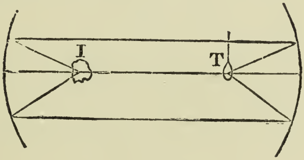


Fig. 183.

One experiment will illustrate these three statements as to radiation. If two similar concave mirrors are arranged facing one the other, and in the principal focus of one is placed a large lump of ice, in the principal focus of the other the bulb of a thermometer, the mercury in the tube of the thermometer falls. The bulb of the thermometer radiates heat: so does the ice in less amount (2). The larger quantity of heat from the thermometer travels in all directions in right lines. As the bulb is in the principal focus, its heat rays fall on the mirror, are reflected horizontally thence, fall on the opposite mirror, and pass to its principal focus where the ice is. The rays of heat from the ice all pursue a similar but reverse course (1). More heat passes from the thermometer bulb than comes to it, and its temperature falls until it has the same temperature as the ice, 0° (3).

The phenomena of dew depend upon the principles of radiation. During the day the earth receives heat from the sun. The air in contact with the earth is therefore hot, and can contain much moisture. When the night-time comes, the earth radiates heat uncompensated as in the day by the greater radiation of the sun. The earth becomes cool; the air in contact with it becomes cool, and can no longer maintain the moisture present in it. This is deposited therefore as dew on the most rapidly-radiating points, such as the ends of grass blades. A tarpaulin spread over the ground, or nature's tarpaulin of clouds, will reflect the radiating rays of heat, and under tarpaulin or clouds dew is not deposited.

APPENDIX I.

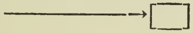
SOLUTIONS TO EXERCISES.

Page 14.

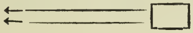
1. (*a*) A mussel breathes by gills. (*b*) The fact of gravitation, or the attraction of matter at a distance; *e.g.*, the sun and earth. The fact of adhesion, or the attraction between unlike bodies near together; *e.g.*, ink on paper.

2. Two masses in motion strike against each other, and heat is evolved. Heat is applied to mercury, and it expands.

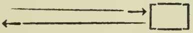
3. (*a*)



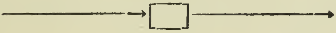
(*b*)



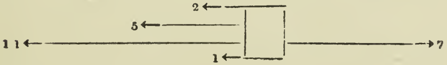
(*c*)



(*d*)

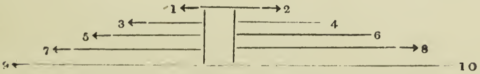


4.



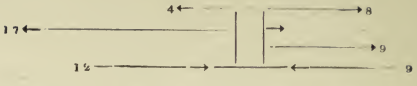
Resultant of $2+5+11+1=19$. $19-7=12$;
∴ 12 must be added.

5.



$1+3+5+7+9=25$. $2+4+6+8+10=30$;
∴ resultant $=5$, pulling in direction of even numbers.

6.



$$4+17-12=9. \qquad 8+1+9-9=9;$$

$\therefore$ body is in equilibrium.

7. The resultant in the first case equals the sum of the forces;

$$\therefore\ 19+x=28. \qquad x=28-19. \qquad x=9.$$

The resultant in the second case equals the difference between the forces;

$$\therefore\ x-29=28. \qquad x=28+19. \qquad x=47.$$

8. $2+5+8+11=26$, aided by -1; $\therefore$ 25;
$\therefore$ on other side 25 required, in arithmetical progression.

$$S=\{2a+(n-1)d\}\frac{n}{2}=\{2+(n-1)2\}\frac{n}{2};$$

$$\therefore\ 25=\{2+2n-2\}\frac{n}{2}$$

$$=n^2;\ \therefore\ n=5.$$

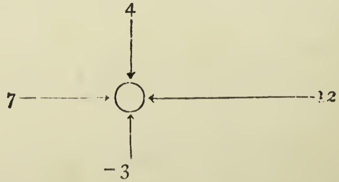
5 forces required, $1+3+5+7+9=25$.

9. $100-7=93$ to be balanced.

$$s=a\frac{(r^n-1)}{r-1};$$

$$\therefore\ 93=\frac{a(32-1)}{2-1}=\frac{32a-a}{1}=31a;\ \therefore\ a=3.$$

10.



Add 7 to − 3, then 4 will be equal and opposite to 4.
" 5 " 7 " 12 " " " 12

Page 22.

1. 16 and 20.

2. 33 and $\sqrt{1530}$ grams.

3. Each $= \dfrac{20}{\sqrt{3 \cdot 414}}$ grams.

4. R at $90^\circ = 5$ grams, at $30^\circ = \sqrt{45 \cdot 784}$, at $150^\circ = \sqrt{4 \cdot 216}$, at $180^\circ = 1$ gram.

5. $\sqrt{114 \cdot 58}$ grams.

6. $1 : \dfrac{\sqrt{3}}{4} (\sqrt{5} - 1)$.

Page 26

1. $5\sqrt{50}$ grams.
2. Force $= 10\sqrt{3}$ grams. Pressure $= 20$ grams.
3. $3\sqrt{2}$ decimetres. $4\sqrt{2}$ grams.
4. 12 grams.
5. 12 grams in longer cord, 16 in shorter.
6. $\dfrac{1100\sqrt{3}}{3}$ grams.

Page 29.

1. A force equal to twice one of the original forces. 120°, 120°, 180°.

2. $\sqrt{147}$.

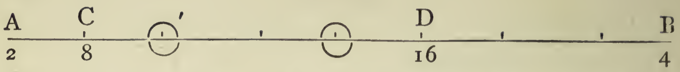
3. $2\sqrt{(5 - 2\sqrt{3})}$

Page 32.

1. (*a*) Forces not applied at same point; two forces of 25 and 17 will be required, acting at points between and opposite to the first and second pair of forces respectively.

(*b*) Forces applied at same point; a force $= 8$ grams will be required, pulling due south, and acting at same point as other forces.

2.



Moment of 2 grams about o′ = 2 × 2 = 4; of 8 grams = 8 × 1 = 8.
Moment of 16 grams about o′ = 16 × 3 = 48; of 4 grams = 4 × 6 = 24.

3. If both act downward, moments are as $3\sqrt{3}$ and 2.

Page 38.

1. Magnitude, 10 grams. Position coincident with the weight of 3 grams or 2 decimetres from fixed end.

2. 160 grams.

3. 10 grams at a distance of 3 decimetres, 14 grams at a distance of 5 decimetres.

4. Forces = 40 grams and 30 grams. Distance between them, 1 decimetre.

5. Between the 5 and 4 grams, distance = 2 decimetres: between the 4 and 3 grams, 3 decimetres; between the 3 and 2 grams, 4 decimetres; between the 2 and 1 grams, 5 decimetres.

6. The forces are 6, 8, 12, 6. The distance of their resultant is $6\frac{7}{8}$ decimetres from one end.

7. 8 decimetres.

8. 20 pounds.

Page 51.

1. If the rod be divided into unit parts whereof the first portion of the rod (to the right) contains r, the second portion s, and the third portion t, the C. G. is

$$\frac{m[2s+2t+r]+n[2t+s]+pt}{2[p+n+m]},$$

such units from the left extremity of the rod.

2. 2 grams. 3. $2\frac{11}{23}$ decimetres from one end.

4. 4 and 8 grams.

5. $\frac{\sqrt{3}}{2}$ decimetres below a point in the bar $3\frac{1}{2}$ decimetres from the middle point.

6. $\frac{1}{2}$ decimetre.

Page 59.

1. $P = \frac{2W}{3} = 40$ kilograms. 2. $5\frac{1}{2}$ kilograms.
3. 60 grams.
4. On A, 10 kilograms; on B, 20.
5. (*a*) 5 metres 6 decimetres; (*b*) 4 metres 8 decimetres; (*c*) the third.

Page 65.

1. 12 grams. 2. $3\frac{5}{41}$ centimetres.
3. Axle, 3 decimetres $1\frac{3}{7}$ centimetres; wheel, 1 metre 5 decimetres $7\frac{1}{7}$ centimetres.
4. 10. 5. $\frac{P}{W} = \frac{2}{29}$. 6. $10\frac{1}{2}$ times.

Page 67.

1. 250. 2. $37\frac{9}{13}$ grams. 3. 40 grams. 4. $2\frac{4}{5}$ decimetres.

Page 87.

1. $18\frac{1}{8}$ grams. 2. 3 grams. 3. $17\frac{1}{2}$ grams.
4. 2, 5, 8, 11, 14, 17 grams. 5. 55 grams. 6. $\frac{13}{14}$ gram.

Page 93.

1. $\frac{120\sqrt{109}}{109}$; $\frac{40\sqrt{109}}{3}$. 2. $33\frac{1}{3}$ grams.
3. $W = 3\sqrt{3}$, $R = 6\sqrt{3}$. 4. $\frac{102\sqrt{11}}{11}$.
5. 8 decimetres. 6. 9 grams.

Page 95.

1. 2 kilograms. 2. 20 decimetres.

Page 98.

1. $17\frac{3}{5}$ grams. 2. $22\frac{1}{2}$ grams. 3. 10.
4. $2\frac{5}{22}$ centimetres. 5. 79·86 grams. 6. $22\frac{3}{11}$ millimetres

Page 102.

1. 3·9 decimetres. 2. 264 centimetres. 3. $\frac{1}{\sqrt{3}} : 2$.
4. 2 centimetres. 5. 1980.

Page 111.

1. 48 grams; $\frac{1}{4}$ decimetre. 2. 1 gram; 40 metres.
3. $1257\frac{1}{7}$ grams. 4. 1 milligram.

Page 125.

1. $8\frac{3}{4}$ grams.
2. 30 centimetres.
3. Base 1000 kilograms; long side 250 kilograms; 125 kilograms.
4. $\frac{1}{8}$ centimetre.
5. 218750 grams due to weight of water; transmitted pressure, $112\frac{7}{11}$ grams; total pressure, $218862\frac{7}{11}$ grams.
6. $899\frac{3}{4}$ grams.

Page 144.

1. 16. 2. $\frac{22}{22\cdot4}$ grams. 3. 100 grams. Nothing.
4. $\frac{1}{2}$. 5. $17\frac{1}{7}$ grams. 6. 48.
7. 30 grams. 8. 3200 grams.

Page 160.

1. 15 centimetres. 2. 3800 millimetres.
3. 25 centimetres. 4. 400 cubic centimetres.
5. $182\frac{1}{4}$ millimetres. 6. $74\frac{26}{27}$ centimetres.
7. 6 : 5. 8. $79\frac{1}{5}$ cubic centimetres.

Page 167.

1. $\frac{1}{2}$ metre. 2. 1764000.

Page 185.

1. Equal; 1 : 10. 2. 4 yards; 20 yards; 32 yards
3. 2 metres; 8 metres; $1\frac{7}{8}$ metres. 4. 5.
5. 55 metres; O; the body is at the starting-point.
6. 100 feet; 20 feet.

Page 203.

1. They are identical and each = 1600.
2. 5th. 3. 490 metres.
4. 49 metres per second; 122·5 metres.
5. $\sqrt{\frac{8\sqrt{3}}{3}}$; $32\sqrt[4]{12}$.
6. $8\sqrt{192}$ feet.
7. 28 feet; 8·575 metres. 8. 1 gram.
9. $16\sqrt{405}$ feet from point of projection.
10. 19·6 metres; $9·8\sqrt{10004}$ metres.

Page 225.

1. $7\frac{1}{17}$ decimetres. 2. $7\frac{1}{2}$ inches behind mirror.
3. 3 centimetres in front of mirror.
4. Image $4\frac{4}{5}$ centimetres from the mirror; diameter $8\frac{8}{11}$ centimetres.

Page 239.

1. 1·5 decimetres. 2. $2\frac{2}{9}$ decimetres.
3. 1·5 decimetres. 4. 5 : 3

Page 248.

1. $9\frac{3}{5}$° R. $53\frac{3}{5}$° F.; 144° R. 356° F.; $-21\frac{3}{5}$° R. $-16\frac{3}{5}$° F.
2. 100° C. 212° F.; $4\frac{2}{7}$° C. $39\frac{5}{7}$° C.; -5° C. 23° F.
3. $11\frac{1}{9}$ C. $8\frac{8}{9}$° R.; 240° C. 192° R.; $-24\frac{4}{9}$° C. $-19\frac{5}{9}$° R.
4. 2100; 70. 5. 1. 6. 10 kilograms.

Page 267.

1. ·000019075.
2. 10 metres nearly.
3. 2553·0898 cubic centimetres.
4. ·001202.
5. $8\frac{160}{273}$ litres.
6. $97\frac{759}{7133}$ cubic centimetres.
7. $28\frac{1}{8}$ cubic centimetres.
8. 20.
9. 4 grams.
10. $\frac{1}{70}$.
11. 45°.
12. Latent heat $18\frac{17}{21}$.
Specific heat $\frac{1}{21}$.

APPENDIX II.

EXERCISES.

Representation of Force.

1. How is a force of 8 kilograms represented graphically on a scale whereon the unit of length is 9 mm., the unit of force being 72 grams?

Ans. 1 metre.

2. If a force equivalent to the weight of 10 grams be graphically represented by a line 1 cm. long, what must be the length of a line that represents a weight of 2 kilograms?

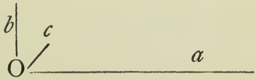
Ans. 2 metres.

3. If on a scale of graphic representation 8 metres represent 3 hectograms, 3 decagrams, 6 grams, and the unit of length be 1 metre, what is the unit of force?

Ans. 42 grams.

4. Represent by means of a diagram on a scale of – to 10 grams the following forces acting on a point O:—(*a*) a horizontal force of 1 hectogram; (*b*) a vertical force of 3 decagrams; (*c*) a force of 20 grams acting at an angle of 30° with the horizontal force.

Ans.



5. If two lines making an angle of 45° with each other represent two forces, and if the two distant extremities of the lines

be joined, and the joining line make an angle of 90° with one of the other two and be 7 cm. long, find the values of the forces represented. (Unit of length 1 mm., unit of force 1 gram.)

Ans. The force with whose line of direction the joining line makes angle 90° = 70 grams; the other = $70\sqrt{2}$.

Parallelogram of Forces.

N.B.— $\sqrt{3} = 1{\cdot}732$, $\sqrt{2} = 1{\cdot}414$.

1. Two equal forces act at right angles one to the other. Find the value of each in grams if the resultant be represented on a scale whereon 1 dcm. corresponds with 1 gram by a line $2\frac{1}{2}$ cm. long.

Ans. $\dfrac{1}{4\sqrt{2}}$ gram.

2. The resultant of two forces at 90° = 10 kilograms, the lesser force = 6 kilograms. Find the value of the greater.

Ans. 8 kilograms.

3. A force of 4 kilograms acts at an angle of 45° to another of 1414 grams. What is the value of the resultant?

Ans. $1000\sqrt{26}$.

4. If the resultant of two equal forces acting on the same point at an angle of 45° = 2·828 kilograms, what is the value of each force?

Ans. $2\sqrt{2-\sqrt{2}}$.

5. Two forces acting on a point at an angle of 30° are respectively represented by $4\sqrt{3}$ and $\sqrt{613}-6$ kilograms. Find the value of their resultant.

Ans. 25 kilograms.

6. Two forces act on a point at an angle of 30°. Find a general expression for the value of the resultant when the horizontal force = m and the inclined force = n.

Ans. $\sqrt{m^2+n^2+mn\sqrt{3}}$.

7. If two forces acting on a point at an angle of 120° are

equivalent respectively to weights of 1 gram and 3 grams respectively, find the value of the resultant.

Ans. $\sqrt{7}$ grams.

8. If two forces act at an angle of 60° with one another, and their ratio is 1 : 2, find their respective values if their resultant $= \sqrt{112}$.

Ans. 4 and 8.

9. Two forces act at an angle of 150°. The value of one of them is 6 grams. What must be the value of the other that their resultant may be equal to that of two forces of 5 and 7 grams respectively acting at an angle of 120°?

Ans. $(3 + \sqrt{10})\sqrt{3}$.

10. With two forces acting at an angle of 135°, one whereof $= \sqrt{2}$ and the resultant $= 3$, find the other force.

Ans. $2\sqrt{2} + 1$.

11. The ratio of one of two forces acting at right angles is to the resultant as 3 : 5. Find the ratio of the two forces one to the other.

Ans. 3 : 4.

12. The ratio of one of two forces at an angle of 120° to the other is 1 : 2. Find the ratio of the resultant to the lesser force.

Ans. $\sqrt{3} : 1$.

Triangle of Forces.

1. Down a perfectly smooth rod inclined at an angle of 45° with the vertical slides a ball weighing 4 grams. Determine the force necessary to be applied to it in a horizontal direction to keep it at rest. What is the pressure upon the rod?

Ans. 4 grams, $4\sqrt{2}$ grams.

2. With a similar arrangement to that in the former question, but the angle $= 30°$, find the weight of a ball kept at rest by a force of $\sqrt{3}$ grams acting parallel to the rod.

Ans. 2 grams.

3. Along a cord whose two ends are fixed to points in the same horizontal line slides a weight. If the two fixed points are 1 metre apart, what is the length of the cord? If the tension in the cord $= 5\sqrt{2}$ grams, what is the weight? The two parts of the cord are at right angles.

Ans. $10\sqrt{2}$ decimetres, 10 grams.

4. With a similar arrangement to that in the preceding question, but the angle between the two parts of the cord 120°, show that the tension in the cord is equal to the weight, and that if the length of the cord be 8 decimetres, the distance between the two fixed points is $4\sqrt{3}$ decimetres.

5. A weight of 40 grams hangs by a vertical string that passes over a fixed pulley and has a weight of 10 grams fixed to its other extremity. A second cord is attached to the weight, and the latter is drawn by aid of this second cord into such a position that the two cords make an angle of 90° one with the other. Determine the tension in the second cord.

Ans. $10\sqrt{15}$ grams.

6. Two cords of total length 70 centimetres are fastened by their extremities to two points in the same horizontal line, and 5 decimetres apart. A kilogram weight is attached to the other ends of the two cords. Determine the exact length of each cord and the tension in each, if the cords are at right angles.

Ans. 30 and 40 centimetres. Tension in short cord, 800 grams; in long cord, 600 grams.

Action of Three or more Forces on a Point.

1. Four forces of value respectively 1, 2, 3, 4 grams act upon a point, the angle contained between successive pairs being 90°. Find the value of their resultant.

Ans. $2\sqrt{2}$.

2. Forces of 6, 6, $2\sqrt{7}$ act on a point. Angle between the two equal forces $= 90°$, angle between one of them and $2\sqrt{7}$ $= 45°$. Find the value of their resultant.

Ans. 10.

3. Three forces, each of 4 grams, with angles of 60° between their directions, act on a point. Show that their resultant is equal to twice one of them.

4. Forces taken in order $= 4, 7, 5, 1, 10, 13, 5, 7$ grams act on a point, the angles between each pair of forces being 45°. Find the magnitude and position of their resultant.

Ans. $3\sqrt{12}$, along the line of the force of 13 grams.

5. Two forces $= 4$ and 8 grams act at an angle of 120° with each other. What must be the value of a third force acting exactly opposite to the first, that the resultant of the three may be $8\sqrt{3}$?

Ans. 12 grams.

Moments.

Rods are supposed weightless unless their weight is given, and uniform unless stated to be otherwise.

1. Compare the moments about the fixed end of a bar 3 metres long of forces of 10 grams, acting at a point 10 decimetres from the fixed end, 4 grams acting in the middle of the bar, $4\sqrt{2}$ grams acting at the remote end, if the first two forces act vertically downwards and the third act downwards at an angle of 45° with the bar.

Ans. 5, 3, 6.

2. To one end of a bar 8 decimetres long is affixed a weight of 8 grams. What must be the value of two equal forces acting upwards at the other end and at the middle of the bar to keep the latter in a horizontal position, if it be fixed at a point 2 decimetres from the 8 grams?

Ans. 2 grams.

3. A bar 2 metres in length has equal weights hung from its two ends. 5 decimetres from one end is applied a force $= 10$ grams acting upwards. 4 decimetres from the other end hangs a weight of 2 grams, and at the same point acts upwards a force $= 3\frac{4}{5}$ grams. The bar balances about a point 1 decimetre nearer the centre of the bar than the point of action of the 10 grams. Find the weights at the ends.

Ans. Each $= 1$ gram.

4. To the two extremities of a bar are applied forces each = 3 grams, one acting upwards, the other downwards. At the middle point is suspended a weight of 5 grams; half a decimetre on that side of the middle point nearer the end where the force acts downwards hangs a weight of 26 grams. The bar balances about a point one-third of its length from that end. Find the length of the bar.

Ans. 6 decimetres.

5. What is the weight of a uniform bar one foot in length if, when a weight of 4 grams hangs from one end, the bar balances about a point one inch from its centre?

Ans. 20 grams.

6. A uniform bar of weight 4 decagrams and length $1\frac{4}{5}$ metres has a weight of $15\frac{1}{2}$ grams hung 3 decimetres from one end. What weight hung at that end, if one-half such a weight be also hung at the other end of the bar, will cause the bar to balance about a point 7 decimetres from the extremity to which the heavier weight is affixed?

Ans. 12 grams.

7. If a uniform bar weighing 12 grams, and of length 1 yard, have 7 grams attached to one end and 4 grams to a point 1 inch from the other end, what upward force acting at the free end of the bar will keep it in equilibrium when the fixed point is $4\frac{10}{21}$ inches from the centre of the bar towards the end where the 7 grams is suspended?

Ans. 2 grams.

8. A bar weighing 10 grams balances about a point 3 decimetres from one end. What weight must be attached to the other end that it may balance about a point one decimetre from this end, if the whole length be $\frac{9}{10}$ of a metre?

Ans. 50 grams.

9. If from one end of a rod one metre long depends a weight of 8 grams, what force acting downwards at the other end at an angle of 60° with the rod will enable the rod to balance about its middle point?

Ans. $\dfrac{16\sqrt{3}}{3}$.

10. Two forces act at the extremities of a bar 1·2 metres in length. One of them $= 10\sqrt{2}$ grams acts at an angle of 45° with the bar. The other acts at an angle of 30° with the bar. Find its value if the point of support be 4 decimetres from the point of application of the former force.

Ans. 10 grams.

11. A weight of 5 grams is affixed to one end of a rod 8 feet long. At the other end acts downwards a force of 19 grams at an angle of 120 degrees with the rod. Determine the weight of the rod if the point of support be two feet from the end whereat the 19 grams acts. Also determine the amount of the horizontal traction on the fixed point.

Ans. $13\frac{1}{2}$ grams, $\frac{19\sqrt{3}}{2}$ grams.

12. A rod is bent so that its two parts are at right angles. From one end hangs a weight of 4 grams, from the other one of 6 grams. The rod now balances about the point of flexure in such a way that a vertical line through that point makes angles of 60° and 30° respectively with the parts of the rod carrying the 4 and the 6 grams. Compare the lengths of the two parts of the rod.

Ans. $\sqrt{3} : 2$.

13. A rod 2 metres long, of weight 4 grams, balances about a point $\frac{1}{4}$ of its length from one end. At that end a force $= 12$ grams acts downwards, making an angle of 30° with the rod. At the other end a force $= 4\sqrt{2}$ grams acts downwards at an angle of 45° with the rod. What force must be introduced at a point midway between the last and the point of support, if that point of support be $\frac{1}{4}$ of the rod's length from the end whereat the $4\sqrt{2}$ grams acts, and the introduced force acts downwards and makes an angle 30° with the bar? The introduced force acts towards the same direction as the $4\sqrt{2}$.

Ans. 88 grams.

Centre of Gravity.

1. Weights of 7, 3, 4, 9, 1 grams are suspended at equal intervals along a weightless rod 16 decimetres long, the 7 and the 1 being

at the two extremities. Where is the centre of gravity of the system?

Ans. 7 decimetres from the 7 grams end.

2. What weight must be attached to the middle of a bar 2·6 metres long, that when weights of 12 and 9 grams respectively are attached to the two extremities, the centre of gravity of the whole may be $1\frac{11}{15}$ decimetres from the middle point?

Ans. $1\frac{1}{2}$ grams.

3. Find how far the centre of gravity of an equilateral uniform triangle of side 1 foot is from the points of bisection of each of its three sides. R

Ans. $2\sqrt{3}$ inches.

4. Show that the centre of gravity of a right-angled triangle whose base $= 3$ and height $= 4$ is $\frac{\sqrt{73}}{6}$ from the centre of the base measured along the line joining that centre and the opposite angle.

5. To an isosceles right-angled triangle is attached a weight = that of the triangle. If the point of attachment be the right angle and the equal sides of the triangle are $6\sqrt{2}$ centimetres long, find the distance of the centre of gravity of the arrangement from the right angle. R

Ans. 2 centimetres.

6. An isosceles triangle having one of its angles = 120°, has the side opposite to that angle $12\sqrt{3}$ centimetres long. At the two lesser angles are affixed weights each = one-half the weight of the triangle. Find the distance of the centre of gravity of the whole arrangement from the angle of 120°.

Ans. 5 centimetres.

7. The upper half of a square 8 decimetres in the side weighs 3 grams, the lower half weighs 1 gram. Find the distance of the centre of gravity of the whole square from the centre of figure. Each half is throughout uniform.

Ans. 1 decimetre above centre of figure.

8. A square 6 inches in the side consists of four equal triangles.

The two above and below weigh respectively 2 and 6 grams; the two lateral ones weigh respectively 2 and 6 grams. Determine the position of the centre of gravity.

Ans. $\sqrt{2}$ inches from the centre of figure along one of the diagonals.

9. A rectangle weighing 10 grams, and having sides 30 and $10\sqrt{7}$ centimetres long, has weights of 5 grams placed at the centre points of two adjacent sides. Where is the centre of gravity of the system?

Ans. 5 centimetres from the centre of figure.

10. A uniform bar of weight 4 grams has a uniform equilateral triangle attached to it so that its base coincides with the last $6\sqrt{3}$ centimetres of the bar, and the apex is below the bar. To the opposite end of the bar is attached a uniform circle of radius 3 centimetres, so that the uppermost point of the circle is as far from the centre of the bar as is the middle point of the base of the triangle. Circle and triangle weigh 4 grams each. Find the centre of gravity of bar, triangle, and circle.

Ans. 2 centimetres below centre of bar.

Levers.

1. A bar 9 decimetres in length is used (*a*) as a lever of the first order, (*b*) as one of the second, (*c*) as one of the third. If the fulcrum is 3 decimetres from the Weight in (*a*) and (*b*), and 3 decimetres from the Power in (*c*), find the Power when the Weight is 24 grams.

Ans. (*a*) 12; (*b*) 8; (*c*) 72 grams.

2. Lever of first order. Power 2 grams, Weight 8 times the Power, Power's arm 8 inches. Find the Weight's arm, the length of the lever, and the pressure on the fulcrum.

Ans. 1 inch; 9 inches; 18 grams.

3. Lever of third order. Weight 10, Power $17\frac{1}{2}$, distance between points of application of Power and Weight 3. Find the length of the lever and the pressure on the fulcrum.

Ans. 7; $7\frac{1}{2}$.

4. A bar weighing 6 grams is fixed at one end. If its length

be 2·4 metres, and 1 decimetre beyond its centre, reckoning from the fixed end, a Weight = that of the bar be attached. Find the Power necessary to be applied at the distant end to keep the bar horizontal.

Ans. $6\frac{1}{4}$ grams.

5. A uniform bar 1 foot long is used as a lever of the first order; the fulcrum is 2 inches from the point of attachment of the Weight, and the Power = 16 grams. The same bar is used as a lever of the second order, and the same Weight as before is affixed at a distance of 3 inches from the fulcrum. The Power is now 30 grams. Find the Weight and the weight of the lever.

Ans. 100 grams and 10 grams.

6. What is the weight of a uniform lever 8 decimetres long, if when forces of $16\sqrt{2}$ and $8\frac{2}{3}$ act downwards at angles of 45° and 30° respectively with the lever, the latter balances about a point 2 decimetres from the point of application of the larger force.

Ans. 3 grams.

Wheel and Axle.

1. The Power being 3 grams, the radius of the axle 3 centimetres, the circumference of the wheel $50\frac{2}{7}$ centimetres. Find the Weight.

Ans. 8 grams.

2. If the mechanical advantage in a simple wheel and axle = 11, find the ratio of the diameter of the axle to the circumference of the wheel.

Ans. $1 : 34\frac{4}{7}$.

3. From the circumference of the wheel to that of the axle measures 1 metre. The circumference of the axle = $12\frac{4}{7}$ decimetres. If the Weight be 6 decagrams, what is the Power?

Ans. 1 decagram.

4. In a compound wheel and axle, what is the difference between the radii of the two axles when with wheel of radius 100, 2 balances 150?

Ans. $2\frac{2}{3}$.

5. Radius of wheel = 40 inches. Diameter of larger axle = 3 inches. Circumference of smaller = $3\frac{1}{7}$ inches. Power = 1 gram. Find Weight.

Ans. 20 grams.

Toothed Wheels.

1. From the axle of a toothed wheel having radius 4 centimetres depends a weight of one kilogram. What weight attached to the axle of a second toothed wheel working on the first, and with circumference $251\frac{3}{7}$ centimetres, will sustain it?

Ans. 1 hectogram.

2. If two wheels carrying 400 and 20 teeth respectively work upon each other, and to the axle of the large a Weight of one hectogram be attached, what weight must be attached to the axle of the other to maintain equilibrium?

Ans. 5 grams.

3. The axle of a toothed wheel having circumference $251\frac{3}{7}$ centimetres carries a weight of $\frac{1}{4}$ gram. What must be the diameter of its fellow that 10 grams on the latter may result in equilibrium?

Ans. 3200 centimetres.

4. If in a toothed wheel of radius 10 centimetres there are 110 teeth, and when 5 grams are attached to its axle and $\frac{1}{2}$ gram to that of its fellow, equilibrium obtains, how many teeth are there in the wheel carrying the $\frac{1}{2}$ gram, and what is the distance between the teeth?

Ans. 11; $\frac{4}{7}$ centimetre.

Pulleys.

1. First system of pulleys; 3 moveable pulleys (weightless). Weight = 24 grams; find Power. If the Power = 1 gram; find Weight.

Ans. 3 grams; 8 grams.

2. First system; four moveable pulleys, weighing from below upwards $3\frac{1}{2}$, 4, 5, $4\frac{1}{2}$ grams. Weight = $88\frac{1}{2}$. Find Power.

Ans. $9\frac{3}{4}$ grams.

3. With a Power = 40 grams, and four pulleys at your disposal whose weights are 2, 3, 4, 6 grams respectively, show what different Weights would be supported according as the pulleys are arranged in ascending or descending order of weight from the fixed pulley downwards.

Ans. 294 grams; 296 grams.

4. What number of pulleys at least must be used on the second system if a Power of $\frac{1}{5}$ is to support a Weight of $1\frac{3}{5}$.

Ans. 8.

5. A block containing 6 pulleys whose weights are 1, $1\frac{1}{2}$, 2, $2\frac{1}{2}$, 3, 5 grams, is used as the lower portion of an arrangement of the second kind. What Weight can be attached to it if the whole is to be sustained by a Power of $2\frac{1}{2}$ grams?

Ans. 15 grams.

6. If the weight of the lowest pulley on the third system be 1 gram, and each of the succeeding pulleys is twice as heavy as the next below, what Weight will be sustained if 6 pulleys in all are used?

Ans. 129 grams.

7. Third system. Weight = 145. Power = 3. Number of pulleys, 5. Find the weight of each pulley, if they are all equal in weight.

Ans. 2 grams.

8. Three pulleys arranged according to the first system. Angles made by the two parts of each cord one with another = 60°. If Power = 4 grams, find Weight.

Ans. 6 grams.

9. A man weighing 56 kilograms supports himself by aid of four pulleys on the first system, and afterwards by aid of eight pulleys on the second system. What is the force exerted by him in each case?

Ans. $6222\frac{2}{9}$ grams in each case.

10. The same man supports himself by aid of three moveable pulleys and one fixed one arranged on the first system. The

pulleys weigh from below upwards 1 hectogram, 150 grams, and 2 hectograms. What is the force he exerts?

Ans. $6355\frac{5}{9}$ grams.

Inclined Plane.

1. On an inclined plane rising 2 in 13, rests a mass of 260 grams. What are the Power and the pressure on the plane respectively when (*a*) the former is horizontal, (*b*) it is parallel to the length of the plane?

Ans. $\frac{104\sqrt{165}}{33}$ grams; $\frac{676\sqrt{165}}{33}$ 40 grams. $20\sqrt{165}$.

2. On a plane of length 2·5 metres, a mass weighing 1 hectogram presses with a force of 125 grams when the Power is acting horizontally. Determine the height and the base of the plane.

Ans. 1·5 and 2 metres.

3. A plane with angle 45°. If the pressure on the plane when the Power acts parallel to the length be $2\sqrt{2}$ grams, what are the Power and the Weight?

Ans. $2\sqrt{2}$; 4 grams.

4. A solid equilateral triangle is bisected. On the longest side of one-half of it is placed a mass weighing 10 grams. Find the horizontal Power necessary to keep the mass from falling, and the Resistance of the plane.

Ans. $10\sqrt{3}$ and 20 grams.

5. Show that in the inclined plane, when the Power is parallel to the base, the Resistance of the plane = the resultant of the Power and the Weight.

6. A mass weighing 1 kilogram is supported on a plane whose angle = 30° by two cords. One cord passes over a fixed pulley at the summit of the plane, the other passes horizontally over a second fixed pulley. The same weight is affixed to each cord. Determine its value.

Ans. $2 - \sqrt{3}$ kilograms.

7. Mass 70 grams on a plane rising 3 in 35. A force of four

grams acts parallel to the length of the plane. Find (a) what vertical force must be applied to the mass to keep it at rest, (b) what horizontal force to keep it at rest.

Ans. (a) $23\frac{1}{3}$ grams; (b) $\frac{35\sqrt{19}}{76}$.

8. Two inclined planes of the same height, viz. one decimetre, and of angles 30° and 45°, have two masses resting upon them, and connected by a string passing over a fixed pulley at the summit. Determine the ratio of the masses.

Ans. $\sqrt{2} : 1$.

Wedge.

1. An isosceles wedge of height 1 foot and base 8 inches is driven in with a force $= \sqrt{10}$ kilograms. What resistance will be overcome?

Ans. 10 kilograms.

2. An isosceles wedge of base 1 decimetre is driven in with a force $= 1$ kilogram, and forces asunder two points, the pressure of each point being 4 kilograms. Find its length of side and height.

Ans. 4 decimetres; $\frac{\sqrt{15}}{2}$.

3. Given the height of an equilateral wedge as $8\sqrt{3}$, determine the ratio of Power to Resistance and also length of side.

Ans. 1 : 2; 16.

4. A Resistance of 100 decagrams is overcome by a pressure of 200 grams. If the side of the wedge be half a metre, what are its base and height?

Ans. 2 decimetres; $2\sqrt{6}$ decimetres.

Screw.

1. A screw has 10 threads to the centimetre, and a circumference of $4\frac{4}{5}$ centimetres. The Weight is 12 grams. Find the Power.

Ans. $\frac{1}{4}$ gram.

2. With circumference = 4 centimetres and the Power = $\frac{1}{20}$ of the weight, how many threads are there to a centimetre?

Ans. 5.

3. A lever 8 decimetres long is used to work a screw having 11 threads to the centimetre. With a Power = $\frac{35}{1936}$ gram, find the Weight.

Ans. 1 hectogram.

4. What is the diameter of a screw, if with 24 threads to the foot the mechanical advantage = $12\frac{4}{7}$?

Ans. 2 inches.

5. The radius of a screw is one decimetre. What number of threads are cut in one metre length of it when a Power of 2 supports a Weight of $1257\frac{1}{7}$?

Ans. 1000.

6. Two screws are employed to raise a mass. One has radius 2·1 centimetres and 3 threads to the centimetre; the other has circumference 12 centimetres, and a distance between its threads of $\frac{2}{5}$ centimetre. Which is the better to use?

Ans. The former.

Hydrostatics.

1. Calculate the pressure on the base of a regular cubical vessel, the length of whose edge is 4 decimetres, when filled with water.

Ans. 64 kilograms.

2. If a piston of area 4 square centimetres is pressed into an opening in one of the sides of the vessel in question (1) with a force = 5 grams, what is the increase of pressure (*a*) on the base, (*b*) on the side in which the piston works?

Ans. 2 kilograms; 1995 grams.

3. The diameters of the pistons of a hydraulic press are 3 and 25 centimetres respectively. Upon the top of the larger is placed a mass weighing $69\frac{4}{9}$ grams. What must be placed on the smaller to maintain equilibrium?

Ans. 1 gram.

4. The height of the water in a hydraulic bellows tube is one metre. If the diameter of the bellows plate be 7 decimetres, what weight can be supported?

Ans. 385 kilograms.

5. The circumference of the small piston in a hydraulic press is $\frac{132}{175}$ of the radius of the larger. What is the mechanical advantage of the press?

Ans. $69\frac{4}{9}$.

6. What is the circumference of the plate of a hydraulic bellows, if with 7 decimetres of water in the tube 7920 grams are supported?

Ans. $37\frac{5}{7}$ centimetres.

7. What is the length of a rectangular box 8 centimetres high, if when filled with water and a pressure of 10 grams on the square centimetre transmitted, the whole pressure on the long side = 2340 grams?

Ans. 2 decimetres.

8. A cylinder of radius two decimetres is half filled with water. The pressure on its base is $3142\frac{6}{7}$ grams. What is the length of the cylinder?

Ans. $\frac{1}{2}$ decimetre.

9. A rectangular vessel of depth 7 decimetres, length 1 metre, breadth 5 decimetres, is partly filled with water. Into the shortest side a piston is let, occupying $\frac{1}{4}$ of the side, and a force = 131250 grams applied thereto. The total pressure on the base is 1050 kilograms. What distance is there between the level of the water and the top of the vessel? The piston is completely immersed.

Ans. 1 decimetre.

Specific Gravity.

1. What is the weight of a cubic metre of a substance of specific gravity 2?

Ans. 2000 kilograms.

2. Required the volume of 4 kilograms of a substance of specific gravity $\frac{1}{2}$.

Ans. 8 cubic decimetres.

3. What is the specific gravity of a solid 3·5 cubic centimetres whereof weigh $17\frac{1}{2}$ grams?

Ans. 5.

4. A solid weighs in vacuo 4 grams, in water 3 grams, in a given liquid 2 grams. Find the specific gravity of the solid and of the given liquid.

Ans. 4 and 2.

5. A cubic metre of a solid of specific gravity 1·5 is weighed in water. Find its weight.

Ans. 500 kilograms.

6. A solid of specific gravity 4 weighs 10 grams in a liquid of specific gravity $\frac{1}{4}$. Find its true weight.

Ans. $10\frac{2}{3}$ grams.

7. Three cubic centimetres of a solid body weighed in a liquid of specific gravity 3 weigh 21 grams. What is its specific gravity?

Ans. 10.

8. Three grams of a light solid are attached to 20 grams of a heavy one. The latter alone weigh 15 grams in water; the two together weigh 10 grams. What are the specific gravities of the heavy and the light body?

Ans. 4 and $\frac{3}{8}$.

9. To one gram of a solid lighter than water are attached 12 grams of a solid of specific gravity 4. The two weigh in water 5 grams. Find the specific gravity of the light substance.

Ans. $\frac{1}{5}$.

10. What weight of a metal of specific gravity 6 must be attached to 3 cubic centimetres of wood of specific gravity $\frac{1}{2}$ to make it float in water in any position?

Ans. $1\frac{4}{5}$ grams.

11. With what force do a cubic decimetre of wood of specific gravity $\frac{1}{4}$, and a cubic decimetre of stone of specific gravity 8, tied together, tend to move downwards when placed in a fluid of specific gravity 2?

Ans. A force = 4250 grams.

12. A cubic foot of a body is placed in water. How far does it sink into the water if its specific gravity be $\frac{1}{6}$?

Ans. 2 inches.

13. A cubic decimetre of a solid sinks in water until only 3 centimetres of its height are not immersed. What weight must be placed on the top of the body to sink its summit exactly to the level of the water?

Ans. 300 grams.

14. A powder of specific gravity 4 is placed in a litre of distilled water. The powder and remaining water weigh 1015 grams. What weight of the powder was introduced?

Ans. 20 grams.

15. What must be the specific gravity of a powder whose weight is 4·8 decagrams, if, when introduced into a litre bottle containing fluid of specific gravity 2, the weight of the powder and remaining fluid = 2 kilograms 4 decagrams 4 grams?

Ans. 24.

16. A piece of wood weighing 20 grams is attached to a piece of iron weighing 140 grams, and of specific gravity 7. The two float in water in any position. Find the specific gravity of the wood.

Ans. $\frac{1}{7}$.

Pneumatics.

1. A fluctuation of level = 10 millimetres occurs in a mercurial barometer. What corresponding change would occur in a barometer, the liquid wherein had specific gravity 4?

Ans. $33\frac{3}{4}$ mm.

2. What would be the specific gravity of a liquid that stood at a height of 105·3 centimetres, when the mercurial barometer is at 780 millimetres?

Ans. 10.

3. Twenty cubic centimetres of gas are measured off under normal pressure. The pressure is increased 140 millimetres. What is the volume of the gas?

Ans. $16\frac{8}{9}$ c.c.

4. The pressure is equivalent to 3040 millimetres. It is reduced to 3000 millimetres less. Find the change of volume undergone by 80 cubic centimetres.

Ans. Expansion = 6 cubic decimetres.

5. Three litres of gas are to be reduced to 500 cubic centimetres. If the original pressure is $1\frac{1}{2}$ kilograms, find the final pressure.

Ans. 9 kilograms.

6. A Boyle's tube has a short limb with length = 1 decimetre. Find (*a*) the difference of level of mercury that has been poured into the tube; (*b*) the whole length of mercury in the tube when the mercury in the short limb is 4 centimetres from the closed end.

Ans. 1140 mm.; 1260 mm.

7. What is the length of the short limb of such a tube that when altogether 2400 millimetres' length of mercury have been poured in, the air in the short limb may occupy 2 centimetres' length?

Ans. 8 cm.

8. A fluid of specific gravity 27 is used in a Boyle's tube. The air left in the short limb occupies 2 centimetres' length. The whole length of fluid in the tube = 8400 millimetres. What is the length of the short limb?

Ans. 25 cm.

9. If the receiver of an air-pump have capacity ten litres, and the cylinder be 4 decimetres long and of radius 7 centimetres, what volume of the original air is removed by the second and by the third strokes?

Ans. 6160 c.c., and $2365\frac{11}{25}$ c.c.

Dynamics.

$g = 9{\cdot}8$ metres or 32 feet.

1. With a given acceleration, find in what second a body will pass over a space equivalent to the velocity acquired in $5\frac{1}{2}''$.

Ans. 6th.

2. With what uniform velocity must a body move that the space traversed in 10″ may be equal to that described in the same time by a body moving with acceleration 4?

Ans. 20.

3. Show that with initial velocity in a direction vertically upwards $= g^2$, the time of ascent $= g$.

4. If an acceleration of 12 metres is working in an opposite direction to an initial velocity of 60 metres, find in what number of seconds the moving body will be 96 metres, first on one side, second on the opposite side of the starting point.

Ans. 2″ or 8″; $5 + \sqrt{41}$.

5. Find acceleration that works against initial velocity of 100, if, when the body has moved over 375 metres, the velocity is half the initial velocity.

Ans. 10.

6. With acceleration = 1, determine the ratio of the velocity at end of 4th″ to the space described.

Ans. 1 : 2.

7. How long must a body fall under the attraction of the earth, that its velocity may be 320 yards per second?

Ans. 30″.

8. A ball is allowed to fall from the summit of a tower. It takes $2\frac{1}{2}$″ to reach the ground. Find the height of the tower.

Ans. 100 feet.

9. If a ball falls from the top of a tower $51\frac{121}{160}$ metres high, how long does it take to reach the ground?

Ans. $3\frac{1}{4}$″.

10. A stone falls under the action of the earth's attraction a distance of 44·1 metres. What is its velocity?

Ans. 6 (4·9) m.

11. In what second does a body falling freely pass over $24\frac{1}{2}$ metres?

Ans. 3d.

12. A ball is thrown horizontally with velocity $= 10\sqrt{10}$ metres from the summit of a cliff 49 metres high. Find its distance from the point of projection when it reaches the ground.

Ans. 111·3 m.

13. From the top of a tower 112 yards high a stone is thrown vertically upwards. It reaches the ground in $7''$. What was its velocity?

Ans. 64 feet.

14. From the same tower as in the preceding question, a stone is thrown up simultaneously with the other, but at an angle of 45° with the vertical. The two reach the ground at the same instant. Find velocity.

Ans. $64\sqrt{2}$.

15. From the same point are thrown balls A, B, C. A is thrown vertically down, and in $2''$ has traversed 40 metres; B, downwards at an angle of 60° with the vertical, with velocity $= 20{\cdot}4$ metres; C, horizontally, with velocity $= 10{\cdot}2\sqrt{3}$. Find the relative positions of A, B, C at the end of the $2''$.

Ans. A and B are on the same horizontal line, and $20{\cdot}4\sqrt{3}$ metres apart; B and C on the same vertical line, and 20·4 metres apart.

16. A body is thrown vertically upwards, with velocity $= 10g$. How far and how long will it rise?

Ans. $50g$; $10''$.

17. With what velocity must a body be thrown vertically upwards to reach the ground again in $8\frac{1}{2}''$? How high does it rise?

Ans. $4\frac{1}{2}g$; $10\frac{1}{8}g$.

18. With what velocity must a body be thrown up that it may rise to a height $=$ the space passed over by a body falling freely for $5''$?

Ans. $5g$.

19. A stone falls from a height for 2″. At the moment it begins to fall, another is thrown vertically upwards from the ground, with velocity $= 5g$. The two stones are level at the end of 2″. What is the height from which the former has fallen?

Ans. $12g$.

20. A body falls down an inclined plane rising $\sqrt{15}$ in 8. If the base be $8\frac{23}{40}\sqrt{15}$ metres, find the time of descent and the velocity acquired.

Ans. 4″; $4{\cdot}9\sqrt{15}$.

21. Down an inclined plane of angle 30° a ball falls for 6″. Over what length of the plane has it passed, and through what vertical height has it fallen?

Ans. $9g$; $4{\cdot}5g$.

22. An inclined plane of angle 60° has length $\frac{32\sqrt{3}}{9}$ feet. One ball rolls down it, another is allowed to fall vertically from its summit. How long do they take to reach the ground, and how far apart are they when they do so?

Ans. $\frac{1}{\sqrt{3}}$; $\frac{16\sqrt{3}}{9}$ feet.

23. An Atwood's machine has two equal weights of 4·8 grams. On one is placed a mass of 2 decigrams. How far will the loaded weight move in 1 second and in the 2d second?

Ans. 1 decimetre; 3 decimetres.

24. What must be the mass of two equal weights in Atwood's machine, if one gram placed on one of them causes the other to ascend $\frac{7}{160}$ metre in $\frac{1}{4}$ of a second?

Ans. 3 grams.

25. Weights of 30 grains each. One has 4 grains placed upon it. After $1\frac{1}{2}$″ this last weight is to be removed. Where would you place the removing ring, and how far would the weights move in the next two seconds?

Ans. $2\frac{1}{4}$ feet below level of top of loaded weight; 6 feet.

26. What weight must be placed on the lighter of two masses

of 9 and 8·6 grams that the velocity acquired in 10″ may be 8 metres?

Ans. 2 grams.

27. On one mass weighing 33 milligrams is placed a milligram. What is the lighter mass if when the velocity is 6 feet the space described is 9 feet?

Ans. 30 milligrams.

28. Two weights of $6\frac{1}{2}$ and $7\frac{1}{2}$. On the former is placed a weight 2. At the end of 2″ this last is removed. How far has each weight travelled, and how long will elapse after the removal of the 2 weight ere the weights are temporarily stationary?

Ans. 8 feet; $3\frac{1}{2}''$.

29. The sum of the two weights of an Atwood's machine = 98 grams, their difference = 1 gram. The heavier is at the top of the machine, the lighter at the bottom. They are found to have reversed their positions in $2\sqrt{5}''$. What is the height of the machine? when do the weights pass each other?

Ans. 1 metre; $\sqrt{10}''$.

30. If a pressure $\frac{1}{4}$ communicate motion to a mass $3\frac{1}{5}$, find the space described in 4″. ($g = 32$.)

Ans. 20 feet.

31. What pressure must act on a mass of 100 that when the space described is 4·9, the velocity may be $\frac{49}{50}$ metre?

Ans. 1.

32. On one edge of a horizontal plane 24·5 metres long rests a mass of 2·9 grams. A string runs horizontally from it over a fixed pulley attached to the other edge, and has a mass of 2 grams affixed to its free end. This mass can descend through a hole in the floor for an unlimited distance. The height of the plane is $2\frac{121}{160}$ metres. Find how long the mass on the plane takes to traverse it, and how long elapses ere it reaches the ground.

Ans. $3\frac{1}{2}''$; $4\frac{1}{4}''$.

Optics.

1. If an image formed by a concave mirror is one-half the size of an object 3 decimetres from the mirror, what is the diameter of the mirror?

Ans. 4 decimetres.

2. An object placed in front of a convex mirror of radius 16 centimetres forms behind the mirror an image 64 millimetres distant from the centre of curvature. Determine the position of the object.

Ans. 1·2 centimetres in front of the mirror.

3. If the focal length of a concave mirror be $\frac{4}{5}$ of the distance of an object from it, what are the relative sizes of image and object?

Ans. 4 : 1.

4. The circumference of the circle whereof a concave mirror forms part is $377\frac{1}{7}$ centimetres. Where must an object be placed that the image formed may be erect, and thrice the magnitude of the object?

Ans. 20 centimetres from mirror.

5. 2 decimetres in front of a convex mirror of focal length 3 decimetres, a candle is placed. Determine the position of its image.

Ans. 1·2 decimetres behind the mirror.

6. With a doubly-convex lens of radius 1 decimetre, an object at a distance of 6 decimetres is investigated. Where is the image formed?

Ans. 1 decimetre 2 centimetres from lens.

7. The distances of the image and object from a doubly-concave lens are 2 and 6 decimetres. What is the radius of the lens?

Ans. 6 decimetres.

8. With a doubly-convex lens of focal length $\frac{1}{2}$ decimetre, the image of an object is on the same side as the object, and half as far again from the lens as is the principal focus. Where is the object?

Ans. 3 centimetres from the lens.

9. A real image five times the linear magnitude of the object is thrown on a screen 3 feet from the object. What is the focal length of the lens employed?

Ans. 5 inches.

Heat.

1. Express 30°, $-5°$, $7\frac{2}{9}°$ on the Centigrade scale in terms of Réaumur and Fahrenheit.

Ans. 24° R., 86° F.; $-4°$ R., 23° F.; $5\frac{7}{9}°$ R., 45° F.

2. Express 60°, $-4°$, $-100°$ on the Réaumur scale in terms of Centigrade and Fahrenheit.

Ans. 75° C., 167° F.; $-5°$ C., 23° F.; $-125°$ C., $-193°$ F.

3. Express 41°, 14°, $-4°$ on the Fahrenheit scale in terms of Centigrade and Réaumur.

Ans. 5° C., 4° R.; $-10°$ C., $-8°$ R.; $-20°$ C., $-16°$ R.

4. If the coefficient of expansion of a solid is ·000004, compare the lengths of a bar thereof at 0° and at 1000°.

Ans. 1 : 1·004.

5. Find the coefficient of expansion of a body a bar whereof 5 metres 1 cm. long at 200° is exactly 5 metres long at 0°.

Ans. ·00001.

6. A litre bottle filled with a liquid at 5° is heated to 125°. On being allowed to cool to 5°, only 952 c.c. of liquid are in the bottle. Find the coefficient of expansion. (The expansion of the bottle is supposed to be *nil.*)

Ans. $\frac{1}{2380}$.

7. A liquid of specific gravity 3 is placed in a litre bottle at 0°. The temperature is raised to 80°. The liquid expelled weighs 32 grams at normal. Find the coefficient of expansion of the liquid.

Ans. $\frac{1}{7500}$.

8. Calculate the change of volume of 60 c.c. of gas at normal when the temperature is raised to 312° and the pressure is 840 mm.

Ans. $56\frac{16}{49}$ c.c.

9. 4 litres of gas are measured off at 800 mm. (temperature 234°). Find their volume at normal.

Ans. $2\frac{66}{247}$ litres.

10. If a certain quantity of gas at 4° has the pressure upon it quadrupled, to what temperature must it be raised that its volume may not alter?

Ans. 835°.

11. How many grams of water must be cooled through 70° to melt 7 grams of ice?

Ans. 8.

12. 40 grams of ice and 40 grams of water are mixed at temperatures of 0° and 90° respectively. What is the final temperature?

Ans. 5°.

13. How much ice at 0° must be mixed with a kilogram of water at 95° that the final temperature may be 15°?

Ans. $842\frac{2}{19}$ grams.

14. If the latent heat of a substance is 50, how much ice will be melted by the amount of heat that will melt a hectogram of the substance?

Ans. $62\frac{1}{2}$ grams.

15. If 4 decagrams of a body at 250° take 10 minutes to cool down to $-10°$, and 12 grams of another at 97 take 3 minutes to cool down to $-3°$, compare their specific heats.

Ans. 13 : 5.

16. A ball of metal weighing 4 hectograms at a temperature of 400° is placed in the ice calorimeter; 1 kilogram of ice is melted. Find the specific heat of the metal.

Ans. $\frac{1}{2}$.

17. A cube 3 decimetres long, 2 broad, 9 centimetres thick, of specific heat $\frac{1}{10}$, is placed when at a temperature of 300° in the ice calorimeter. How much ice will be melted?

Ans. 8 kilograms 1 hectogram.

18. 17 grams of water at 40° are mixed with 20 grams of a liquid at 10°, and the resulting temperature is $30\frac{10}{21}°$. What is the specific heat of the liquid?

Ans. $\frac{17}{43}$.

19. 1 gram of ice is melted, and the resulting water is raised to a temperature of 10° by admixture with a hectogram of a body at $12\frac{4}{5}$°. What is the specific heat of the latter?

Ans. $\frac{9}{28}$.

20. 40 grams of a solid at 12° are melted and raised to 14° by admixture with 94 grams of water at 24°. Half a gram of the solid is melted and raised to 27° by admixture with 1 gram of water at 42°. Find the latent and specific heat of the substance, its melting point being 12°.

Ans. $22\frac{1}{2}$; $\frac{1}{2}$.

21. Compare the conductivities of two substances, if through a regular cube 2 cm. in the side of one flows in one hour enough heat to melt one gram of ice, the difference of temperature of the two sides being 6°, and through a cylinder of the other a decimetre long and of radius 7 cm., enough heat flows in 30 minutes to raise the temperature of 3 grams of water from 2° to 5°, the temperature of the two sides being respectively 10° and 6°.

Ans. 616 : 27.

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